THE MATHEMATICAL GAZETTE

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VOL. XXX. FEBRUARY, 1946. No. 288

ANNUAL GENERAL MEETING, 1946.

THE meeting is fixed for Wednesday, 24th April, and Thursday, 25th April, at the Polytechnic, Regent Street, London, W. 1.

On the first day, the business meeting will be held at 10.30 a.m., followed by the Presidential Address by Professor S. Chapman, F.R.S. In the afternoon, there will be a discussion at 2 p.m., and a paper at 4.45 p.m.

On the second day, the morning papers will be at 10 a.m. and 11 a.m., the afternoon discussion at 2 p.m., and the final paper at 4.45 p.m.

Lunch and tea will be available on the premises.

A detailed programme will be circulated in March.

SIGN CONVENTIONS AND CURVATURE.

BY A. ROBSON.

THIS note is intended to supplement Note 1754 and the article on pages 77 to 79 of *Gazette* XXVII. There is some danger that the reader of that note or article may get the impression that the writers are trying to prove some absolute truth about the signs when they are really only concerned with consequences of the definitions and conventions. There is not much difference between a definition and a convention. Often a convention is little more

than an instruction about how a definition is to be interpreted. Neither a definition nor a convention can be right or wrong: it can be appropriate or inappropriate.

The theory of curvature begins with the definition of the curvature κ as $d\psi/ds$, assuming that a preliminary explanation has been given about the meaning of s and ψ. This is much to be preferred to the definition by d^2y/dx^2, and there is no good reason for starting with the reciprocal of the curvature rather than with the curvature itself.

The arc s which is defined as an integral may start life as a signless number, but it is customary and convenient to attach a plus or a minus sign to an arc OP of a curve AOB according as P lies on one or other of the parts OA, OB. When this has been done, a positive direction of the tangent is defined by taking it to be the direction in which s increases. This is needed for the precise definition of ψ. ψ is an angle from an initial line, which is usually OX or $\theta=0$, to the direction of the positive tangent, and is taken to be positive when the rotation is counter-clockwise as in trigonometry. Thus ψ is indeterminate to the extent of an additive constant $2n\pi$. Up to this point, there will be general agreement except for the nihilists who eschew all sign-conventions. The definition and conventions about the arc and curvature are very clearly set out by R. H. Fowler in Cambridge Tract No. 20, which is followed in the present note, except that the details are not always discussed by Fowler as fully as necessary for beginners, because he was writing for university students.

The direction of measurement of s is chosen so that s increases with the parameter or with the independent variable. This implies that s is not always measured in the same direction for a particular curve at a specified point; and it also implies the need on occasion for dividing up the curve into two or more parts for the consideration of the arc and the curvature.

The equations that occur most frequently in elementary work are

$$\{x=f(t),\quad y=g(t)\},\quad y=g(x),\quad x=f(y),\quad r=h(\theta),$$

and the functions f, g, h are one-valued, or, if they are not, branches of the curve given by different values of the functions must be considered separately. As curves generally present themselves in these forms, and not in the form $\psi=f(s)$, it is found to be useful to transform $d\psi/ds$ into shapes suitable for application to these types. The adoption of the sign-conventions secures that the same formulae give the arc or the curvature without ambiguity of sign

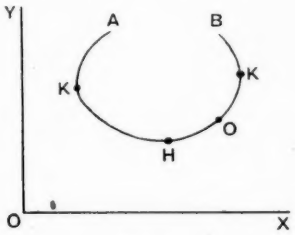


FIG. 1.

for each type of equation separately. In the diagram, however, it is not always true that an arc OP is taken to be positive when P is in OB and negative when it is in OA. For an equation $y=g(x)$ the direction of measurement of s changes at K; for $x=f(y)$ it changes at H; for $r=h(\theta)$ it changes

somewhere else ; and for parametric equations it depends upon the choice of parameter. It may be tempting, but it is not feasible, to make all the different conventions give the same sign for the curvature at every point regardless of the kind of equation that is used. The sign of the curvature is not an intrinsic property of the curve : it depends upon the conventions that are being used.

In simple applications, it often happens that it is more convenient to use $d\psi/ds$ directly than to use one of the formulae derived from it. For example, consider the rectangular hyperbola given by $x=t$, $y=1/t$.

Here $\dot{s}=\sqrt{(\dot{x}^2+\dot{y}^2)}=\sqrt{(1+1/t^4)}$, where, as usual, $\sqrt{a}$ denotes the positive square-root of a. And $\psi=\tan^{-1}(-1/t^2)$, whence $\dot{\psi}=2t/(1+t^4)$. Thus

$$\kappa=d\psi/ds=2t^3/\sqrt{(1+t^4)^3}.$$

It is perhaps worth while to emphasise the need to be clear about $\sqrt{a}$. It should be restricted to the positive value of the square-root, and $a^{\frac{1}{2}}$ can be used to denote $\pm\sqrt{a}$. It is useless to make careful sign-conventions about the curvature and then to use an ambiguous formula for the length of the arc. Even in elementary algebra, it is well to be definite about $\sqrt{a}$.

The use of *ab initio* methods with $d\psi/ds$ economises in sign-conventions, but it does not affect the desirability for clearness about the measurement of s. Consider, for example, the length of the arc of the curve

$$\{x=\cos t\sin^2 t,\quad y=\sin t\,(1+\cos^2 t)\}$$

from $t=0$ to $t=\frac{1}{2}\pi$. The formula for s is

$$\int_0^{\pi/2}\sqrt{(\dot{x}^2+\dot{y}^2)}\,dt \quad \text{and is not} \quad \int_0^{\pi/2}(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}\,dt.$$

This gives

$$\int_0^{\pi/2}\sqrt{(2\cos^2 t-\sin^2 t)^2}\,dt\equiv\int_0^{\alpha}(2\cos^2 t-\sin^2 t)\,dt+\int_{\alpha}^{\pi/2}(\sin^2 t-2\cos^2 t)\,dt,$$

where $\alpha=\tan^{-1}\sqrt{2}$, and so the length of the arc is $\tan^{-1}\sqrt{2}-\frac{1}{4}\pi+\sqrt{2}$.

An implicit equation $F(x,y)=0$ must always be replaced for the calculation of the curvature by an explicit equation. This is explained in Fowler's tract. There is nothing unusual about it. The same thing is done, for example, when questions arise about continuity. But even for the calculation of dy/dx for a particular point of $x^2+y^2=1$, although the formula $dy/dx=-x/y$ can be found without the introduction of roots, it is necessary for the calculation of the gradient at any special point to decide which value of y is to be taken.

How then is the curvature to be discussed for an implicit equation $y^2-x=0$? Normally the equation would be replaced by $\{x=t^2,\ y=t\}$ or by the explicit equation $x=y^2$. But if it is desired to use $y=g(x)$, the curve must be divided into two parts $y=+\sqrt{x}$ and $y=-\sqrt{x}$, and the two parts must be considered separately. With the usual conventions the curvature found for $y=+\sqrt{x}$ is negative and that for $y=-\sqrt{x}$ is positive. If at this point it is asked (as on p. 77 of the *Gazette* article) what is the curvature at the origin, the answer is that neither part of the curve has a curvature at the origin unless an extension is made of the definition. A natural extension is suggested by the usual theory of right- and left-handed differentiation ; and in this way the value of the " curvature " would be -2 at the origin on $y=+\sqrt{x}$, and $+2$ at the origin on $y=-\sqrt{x}$. There is no scandal in the inequality of these curvatures. The signs do not express an intrinsic property of the curve : but taken in

conjunction with the conventions, as they must be, they both give the same piece of information about the direction of the concavity.

There are some other subsidiary quantities that need to be carefully defined. These arise in connection with polar equations.

The polar coordinate r is not usually restricted to be positive, and sometimes it would be very inconvenient to make that restriction.

ϕ is an angle from the radius-vector to the positive tangent. Combined with the definitions of θ and ψ this gives $\psi = \theta + \phi \pmod{2\pi}$. This is a good illustration of a deduction made directly from the definitions avoiding the use of any special figure. No proof is satisfactory which relies on a figure; though a figure may be invaluable to help the imagination of the learner or to stimulate the inventive powers of the researcher.

ϕ can be restricted to the range from 0 (inclusive) to 2π (exclusive), though this is not an essential convention.

The direction of the positive normal is defined to be $+\frac{1}{2}\pi$ from the positive tangent, again with the counter-clockwise convention.

p is the perpendicular distance from the pole to the tangent measured in the direction of the negative normal. It can be proved that $p = r \sin \phi = r^2 d\theta/ds$, and as s is taken to increase with θ, p is always positive. It can also be proved that $dp/d\psi = r \cos \phi$ and that $1/\kappa = r dr/dp = p + d^2p/d\psi^2$. This last result is proved in Note 1754 in a very instructive way; and if it is objected that the method used is not the one by which the other formulae for κ are derived, it may be argued that as $p = f(\psi)$ is essentially an envelope equation, it is very suitable that this formula should be proved by an envelope method. It is interesting to discuss with beginners whether in differentiating

$$x \sin \psi - y \cos \psi = p$$

it is necessary to treat x and y as variables and to cancel the terms

$$dx \sin \psi - dy \cos \psi,$$

or whether x and y can be taken as constants.

Up to this point the reciprocal ρ of the curvature κ has been studiously avoided. It must be admitted that it is not without interest in connection with the circle of curvature, the evolute, and the acceleration of a point moving in a curve. This acceleration is found by differentiation with respect to the time of $v\hat{\mathbf{t}}$, where $\hat{\mathbf{t}}$ is a unit vector along the positive tangent, and it is therefore $\dot{v}\hat{\mathbf{t}} + v\dot{\psi}\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector in the positive normal. Hence there is a component acceleration $= v\dot{\psi} = \kappa v^2$ along the positive normal. On pp. 77, 78, of the *Gazette* the fact that the acceleration is v^2/ρ along the inwards normal is cited as an argument to show that ρ must be signless. The warning ought to have been added that it is impossible to prove that the acceleration is v^2/ρ along the inwards normal without assuming that ρ is signless. Perhaps it is inevitable that mathematicians will regard the component as κv^2 along the positive normal, and physicists will regard it as v^2/ρ along the inwards normal: they will both be right.

If it is desired to avoid the use of a negative radius of a circle, the circle of curvature may be defined for a point P of a curve as a circle through P with its centre C on the normal so that the displacement from P to C is ρ along the positive normal. In the example (above) about the hyperbola, with the equations chosen, the positive directions of s, the tangent, and the normal are those shown in the first diagram of Fig. 2. If parametric equations $x = 1/t$, $y = t$ had been used, the positive directions would have been those in the second diagram. In both cases PC is ρ along the positive normal; ρ is positive in (i) and negative in (ii).

Another example in which some care is needed is the following: A heavy ring falls from rest at the origin on a smooth wire in the form of the curve

FIG. 2.

$y = x^3$ with the positive y-axis vertically upwards. Find whether the force exerted by the wire on the ring becomes instantaneously zero.

The forces are those marked on the diagram where R is along the positive normal. The curvature $= \kappa = 6x/\sqrt{(1+9x^4)^3}$; $\tan\psi = 3x^2$, and so

$$\cos\psi = 1/\sqrt{(1+9x^4)}; \quad \text{and} \quad v^2 = 2g(-y) = -2gx^3.$$

The equation of motion is

$$R - mg\cos\psi = m\kappa v^2.$$

κv^2 is along the positive normal; and that is why the force must be taken along the positive normal, and therefore must not be $mg\cos\psi - R$. R changes from positive to negative when $1 + 9x^4 = 12x^4$, *i.e.* when x passes through the value $-1/\sqrt[4]{3}$.

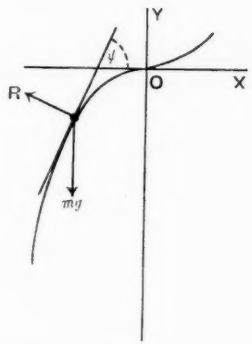


FIG. 3.

A. R.

PLANES OF CIRCULAR SECTION OF A CONE.

By R. T. Robinson.

1. If BPC, DPE are circular sections of the cone vertex A, a sphere can be drawn containing these two sections, and the cone will touch the sphere at the points P, P_1 where the line of intersection of the two planes meets the sphere. Thus $AP^2 = AD \,.\, AB$. Taking BPC as the base of the cone, if AC is the least and AB the greatest of the generating lines drawn from A to the base, ABC will be a principal section of the cone and the planes of circular section of the cone are perpendicular to the plane ABC. If AF is perpendicular to DE and AH perpendicular to BC,

$$AP^2 = AF \,.\, AH \sin B \sin C\,;$$

that is, AP^2 is the product of the perpendiculars drawn from P to the planes through the vertex A parallel to the planes of circular section, divided by $\sin B \sin C$.

2. *To show that if a cone stands on one of its circular sections as base, the eccentricity of the principal section of the cone perpendicular to the axis is equal to the difference between the greatest and least generators of the cone divided by the diameter of the circular base.*

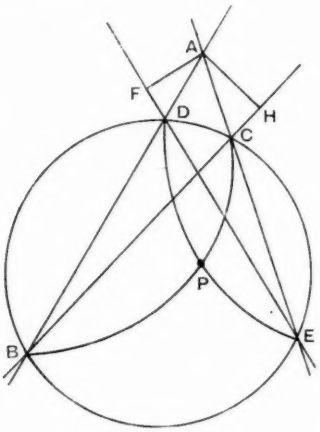


Fig. 1.

For example, in Fig. 1 the eccentricity of the principal section of the cone $ABCDE$ can be written as $(AB - AC)/BC$ or $(AE - AD)/DE$.

If a sphere is drawn through A containing the base BPC and ABC is a principal section of the cone, the tangent plane at A to the sphere is parallel to the other plane of circular section, and therefore if O is the centre of the sphere and if AM is perpendicular to BC, the straight lines AO, AM are the focal lines of the reciprocal cone. If the perpendicular from O on to the plane BPC meets the sphere in D, AD is the axis of the cone.

To show that the eccentricity e of the section of the cone perpendicular to the axis is equal to $(c - b)/a$: if H is the point of intersection of AD and BC

and if KPP_1 is the plane perpendicular to AH, PH is perpendicular to the plane ABC and

$$PH^2 = BH \,.\, HC = AH^2 \,.\, \sin^2 \tfrac{1}{2}A / \sin B \sin C,$$
$$KH^2 = AH^2 \,.\, \tan^2 \tfrac{1}{2}A.$$

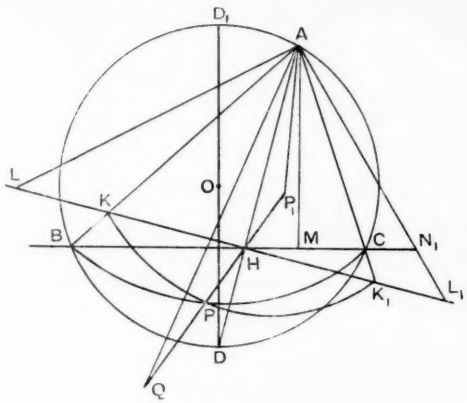


FIG. 2.

Thus $PH^2 - KH^2 = AH^2 \sin^2 \tfrac{1}{2}A \,(\cos^2 \tfrac{1}{2}A - \sin B \sin C)/\cos^2 \tfrac{1}{2}A \sin B \sin C$
$= AH^2 \tan^2 \tfrac{1}{2}A \sin^2 \tfrac{1}{2}(C - B)/\sin B \sin C,$

which is positive.

Thus PH is the major semi-axis and KH the minor semi-axis of the section. Hence

$$e^2 = (PH^2 - KH^2)/PH^2 = \sin^2 \tfrac{1}{2}(C - B)/\cos^2 \tfrac{1}{2}A,$$

and $$e = (\sin C - \sin B)/\sin A = (c - b)/a.$$

To show that the eccentricity of the section perpendicular to the axis of the reciprocal cone is also equal to $(c - b)/a$,

let AL perpendicular to AC meet KK_1 in L,
AL_1 perpendicular to AB meet KK_1 in L_1,
AQ perpendicular to AP_1 meet HP in Q.

Then $AH^2 = P_1H \,.\, HQ = HP \,.\, HQ,$

that is, $HQ^2 = AH^4/HP^2 = AH^2 \sin B \cos C/\sin^2 \tfrac{1}{2}A,$

and $HL = AH \cos \tfrac{1}{2}A.$

Thus $e = (HL^2 - HQ^2)/HL^2 = (\cos^2 \tfrac{1}{2}A - \sin B \sin C)/\cos^2 \tfrac{1}{2}A$
$= \sin^2 \tfrac{1}{2}(C - B)/\cos^2 \tfrac{1}{2}A,$

and as before, $e = (c - b)/a$.

3. To show that M is a focus of the section of the reciprocal cone by the plane BPC, let this plane meet AL, AL_1 in N and N_1.

The point Q is on this section, and if α, β are the major and minor semi-axes of the section,

$$\beta^2/\alpha^2 = QH^2/NH \,.\, N_1H = \frac{AH^2 \sin B \sin C}{\sin^2 \tfrac{1}{2}A} \cdot \frac{\cos B \cos C}{AH^2 \cos^2 \tfrac{1}{2}A},$$

and $$\alpha = \tfrac{1}{2}(NH + N_1H) = \tfrac{1}{2}AH \cos \tfrac{1}{2}A \,(\cos B + \cos C)/\cos B \cos C.$$

Hence
$$\beta^2 = \frac{AH^2}{4} \cdot \frac{\sin B \sin C}{\cos B \cos C} \cdot \frac{(\cos B + \cos C)^2}{\sin^2 \tfrac{1}{2}A}$$
$$= AH^2 \,.\, \cos^2 \tfrac{1}{2}(C - B) \tan B \tan C$$
$$= AM^2 \tan B \tan C$$
$$= NM \,.\, N_1M\,;$$

that is, M is a focus of the section.

4. To show that O is a focus of the section LL_1 of the reciprocal cone drawn through O perpendicular to the radius OA, let the perpendicular from H to the plane ABC meet the circular section NPN_1P_1 of the cone in P, P_1 and the section LL_1 in Q, Q_1; then ABC and PAP_1 being principal sections, P_1AQ is a right-angle and

$$AH^2 = HP_1 \,.\, HQ = HP \,.\, HQ,$$

and
$$HP^2 = NH \,.\, N_1H.$$

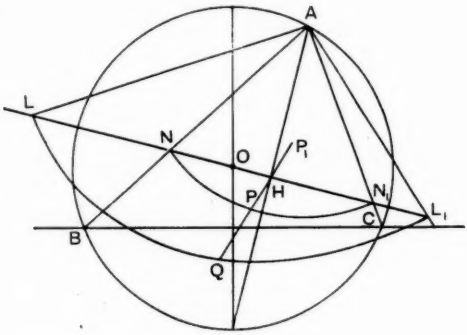


FIG. 3.

But $NH/AH = \sin \tfrac{1}{2}A/\sin C$ and $N_1H/AH = \sin \tfrac{1}{2}A/\sin B$.

Thus
$$HP^2 = AH^2 \sin^2 \tfrac{1}{2}A/\sin B \sin C,$$

and
$$HQ^2 = AH^2 \sin B \sin C/\sin^2 \tfrac{1}{2}A.$$

Also $LH = AH \cos \tfrac{1}{2}A/\cos B$, $L_1H = AH \cos \tfrac{1}{2}A/\cos C$.

Thus if α, β are the major and minor semi-axes of the section LQL_1,

$$\beta^2/\alpha^2 = HQ^2/LH \,.\, L_1H = 4 \sin B \sin C \cos B \cos C/\sin^2 A$$

and
$$2\alpha = LO + L_1O = R(\tan B + \tan C) = R \sin A/\cos B \cos C.$$

Thence
$$\beta^2 = R^2 \tan B \tan C = OL \,.\, OL_1,$$

and hence O is a focus of the section.

5. If in Fig. 2 the bisectors AD_1, AD of the angle A are taken as axes of x and z, and the straight line through A perpendicular to the plane ABC as the axis of y, the equation of the plane through A parallel to the circular section BPC is

$$z\cos\tfrac{1}{2}(C-B)+x\sin\tfrac{1}{2}(C-B)=0,$$

and the equation of the tangent plane at A to the sphere, which is parallel to the other set of circular sections, is

$$z\cos\tfrac{1}{2}(C-B)-x\sin\tfrac{1}{2}(C-B)=0.$$

If P is any point on the cone, the product of the perpendiculars from P to these two planes is $AP^2\sin B\sin C$; that is,

$$(x^2+y^2+z^2)\sin B\sin C=z^2\cos^2\tfrac{1}{2}(C-B)-x^2\sin^2\tfrac{1}{2}(C-B),$$

or

$$x^2\cos^2\tfrac{1}{2}A+y^2\sin B\sin C-z^2\sin^2\tfrac{1}{2}A=0;$$

this is the equation of the cone. The equation of the reciprocal cone is

$$\frac{x^2}{\cos^2\frac{1}{2}A}+\frac{y^2}{\sin B\sin C}-\frac{z^2}{\sin^2\frac{1}{2}A}=0.$$

The last equation can be written

$$x^2+y^2+z^2+y^2\frac{\sin^2\frac{1}{2}(C-B)}{\sin B\sin C}-\frac{z^2}{\sin^2\frac{1}{2}A}=0.$$

Hence the planes of circular section of the reciprocal cone are

$$y(\sin C-\sin B)\pm 2z\sqrt{(\sin B\sin C)}=0,$$

or

$$y(c-b)\pm 2z\sqrt{(bc)}=0,$$

and the focal lines of the original cone are given by

$$x=0,\quad y/(c-b)=\pm z/2\sqrt{(bc)}.$$

6. If the equation of a cone is

$$S\equiv ax^2+by^2+cz^2+2hxy+2gzx+2fyz=0,$$

it can be written in the form

$$S-\theta(x^2+y^2+z^2)+\theta\,.\,OP^2=0,$$

where O is the vertex of the cone, P any point on the cone, and θ a constant; that is to say, it can be written in the form

$$(a-\theta)x^2+(b-\theta)y^2+(c-\theta)z^2+2hxy+2gzx+2fyz+\theta\,.\,OP^2=0.$$

In a cone, OP^2 has a constant ratio to the product of the perpendiculars drawn from P to the planes through the vertex drawn parallel to the planes of circular section. The terms independent of OP^2 in the equation will form the equation of two planes if

$$\begin{vmatrix} a-\theta, & h, & g \\ h, & b-\theta, & f \\ g, & f, & c-\theta \end{vmatrix}=0,$$

that is, if θ is a root of the equation

$$\theta^3-\theta^2(a+b+c)+\theta(bc+ca+ab-a^2-b^2-c^2)-\Delta=0.$$

Therefore when θ has one of these values the equation of the planes drawn through the vertex parallel to the planes of circular section is

$$(a-\theta)x^2+(b-\theta)y^2+(c-\theta)z^2+2fyz+2gzx+2hxy=0.$$

7. In quadriplanar coordinates, if in the tetrahedron of reference D is the vertex of the cone, and DA, DB, DC are generating lines of the cone, and P is any point on the cone, then if ABC is a circular section of the cone the equation of the cone is

$$c^2AB\alpha\beta+b^2AC\alpha\gamma+a^2BC\beta\gamma=0.$$

Now
$$\begin{aligned}-9V^2\,.\,DP^2&=\Sigma c^2AB\alpha\beta-3V(d^2A\alpha+e^2B\beta+f^2C\gamma)\\&=\Sigma c^2AB\alpha\beta-(3V-D\delta)(d^2A\alpha+e^2B\beta+f^2C\gamma)\\&=\Sigma c^2AB\alpha\beta-(A\alpha+B\beta+C\gamma)(d^2A\alpha+e^2B\beta+f^2C\gamma);\end{aligned}$$

that is, since P is on the cone,

$$DP^2=(A\alpha+B\beta+C\gamma)(d^2A\alpha+e^2B\beta+f^2C\gamma)/9V^2;$$

but $A\alpha+B\beta+C\gamma=0$ is the equation of the plane through D drawn parallel to the base ABC, and $d^2A\alpha+e^2B\beta+f^2C\gamma=0$ is the equation of the tangent plane at D to the sphere $ABCD$ of radius ρ; hence DP^2 is a multiple of the product of the perpendiculars from P to the planes through D drawn parallel to the planes of circular section, the multiplier being $2\rho D/3V$.

R. T. ROBINSON.

LONDON BRANCH.

REPORT OF MEETINGS.

SIX meetings have been held in 1945. Two afternoons were devoted to a discussion of the Report of the Jeffery Committee on General Schools' Mathematics. "Mathematics in the New Secondary Schools" was discussed at another meeting. Other topics included Plan and Elevation, the Decimalisation of Money, Solution of Equations by Graphs, and the Teaching of Elementary Mechanics. Professor Roberts delivered a Presidential Address on "Mathematics at the Royal Military Academy, Woolwich".

A. J. TAYLOR, *Hon. Sec.*

SHEFFIELD AND DISTRICT BRANCH.

AN enjoyable and representative gathering was held on the afternoon and evening of Friday, 23rd November, 1945, on the occasion of the seventh "birthday" of the Branch.

Twenty members accepted the very kind invitation of Mr. Northeast and his Staff to meet for tea at the City Grammar School, Sheffield; seven other members sent goodwill messages. Following a delightful "pre-war" tea, at a very moderate cost, a short, informal general meeting was held in the Library. Miss Holdsworth presided; the minutes and accounts were accepted without discussion; it was agreed that no subscription should be asked for the year ending December, 1945. The following Officers for 1946 were elected by unanimous vote; *President*, Dr. J. R. Thompson, M.C.; *Vice-Presidents*, Mr. R. R. S. Cox, Mr. A. B. Ward; *Hon. Treasurer*, Mr. J. W. Cowley; *Hon. Secretary*, Mr. C. R. Barwell; *Committee*, Misses Holdsworth and Wilkins, Messrs. Blackman, Blackwell, Davison, G. W. Wilkinson. A party of twelve members then proceeded to the Playhouse to see *The Late Christopher Bean*.

JOHN W. COWLEY, *Hon. Secretary-Treasurer.*

SUCCESSIVE PEDAL TRIANGLES.

By D. G. Taylor.

1. Starting with any triangle $\Delta \equiv ABC$, let $\Delta_1 \equiv A_1B_1C_1$ be its pedal, $\Delta_2 \equiv A_2B_2C_2$ the pedal of Δ_1, and so on indefinitely. Since the circumradius R_n of Δ_n is equal to $\frac{1}{2}R_{n-1}$, we may expect a steady contraction towards some definite limiting point. It is proposed to inquire into the position of this point.

2. If Δ is right-angled at A, the sequence of triangles is finite, and A is the limiting point. We may regard Δ_1 as collapsing to the line AA_1, and Δ_2 as contracting to the point A. The sequence will likewise terminate if a right-angle appears in any triangle Δ_r. But if no right-angle appears at any stage, the sequence will be an infinite one. The case of the equilateral triangle need not detain us.

3. We are thus led to examine the angles of successive triangles.* If Δ is acute-angled, the elements of the pedal are given by

$$\left.\begin{aligned} A_1 &= \pi - 2A, & a_1 &= a\cos A \\ B_1 &= \pi - 2B, & b_1 &= b\cos B \\ C_1 &= \pi - 2C, & c_1 &= c\cos C \end{aligned}\right\}; \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

but if A is obtuse, the formulae are

$$\left.\begin{aligned} A_1 &= 2A - \pi, & a_1 &= -a\cos A \\ B_1 &= 2B, & b_1 &= b\cos B \\ C_1 &= 2C, & c_1 &= c\cos C \end{aligned}\right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

This alternative introduces a complication into the general formulae for the angles of Δ_n. So long as all angles are acute, we have

$$A_1 - \tfrac{1}{3}\pi = (-2)(A - \tfrac{1}{3}\pi), \quad A_n - \tfrac{1}{3}\pi = (-2)^n(A - \tfrac{1}{3}\pi), \quad \ldots\ldots\ldots\ldots(3)$$

and so for B_n, C_n. But this formula itself shows that an obtuse angle is bound to arise sooner or later.

4. To obtain a workable result, let us consider what happens in the *double* step, from Δ to Δ_2. Similar formulae will connect Δ_n, Δ_{n+2}. We may have the following cases :

I. Δ, Δ_1 *both acute-angled* :

$$\left.\begin{aligned} A_2 &= 4A - \pi \\ B_2 &= 4B - \pi \\ C_2 &= 4C - \pi \end{aligned}\right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

II. A *the only obtuse angle in the pair of triangles* :

$$\left.\begin{aligned} A_2 &= \pi - 2A_1 = \pi - 2(2A - \pi) = 3\pi - 4A \\ B_2 &= \pi - 2B_1 = \pi - 4B \\ C_2 &= \pi - 2C_1 = \pi - 4C \end{aligned}\right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

III. A_1 *the only obtuse angle in the pair of triangles* :

$$\left.\begin{aligned} A_2 &= 2A_1 - \pi = 2(\pi - 2A) - \pi = \pi - 4A \\ B_2 &= 2B_1 = 2(\pi - 2B) = 2\pi - 4B \\ C_2 &= 2C_1 = 2(\pi - 2C) = 2\pi - 4C \end{aligned}\right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

* Cp. " Angles of Pedal Triangles ", C. O. Tuckey, *Mathematical Gazette*, XVII, 1933), p. 48. I am indebted to the Editor for this reference.

IV. A, A_1, *both obtuse*:

$$\left.\begin{aligned} A_2 &= 2A_1 - \pi = 2(2A - \pi) - \pi = 4A - 3\pi \\ B_2 &= 2B_1 \quad = \quad 4B \\ C_2 &= 2C_1 \quad = \quad 4C \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

V. *Non-corresponding angles* (*say* A, B_1) *obtuse*:

$$\left.\begin{aligned} A_2 &= 2A_1 = 2(2A - \pi) \quad = 4A - 2\pi \\ B_2 &= 2B_1 - \pi \quad = 4B - \pi \\ C_2 &= 2C_1 \quad = 4C \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

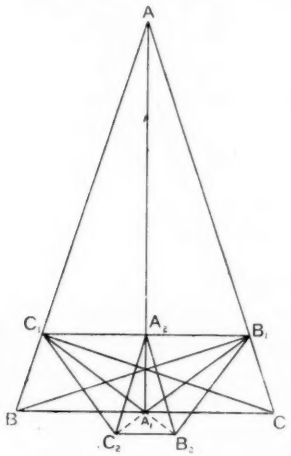
If for the moment we ignore alternative V, and consider the formulae for the angles A_n from (1) to (7), we shall find that, for all values of n, A_n is of the form

$$A_n = \pm\{2^n A - (2k-1)\pi\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

k being the appropriate integer to bring A_n within the range $(0, \pi)$.

5. Let us consider the case of an *isosceles* triangle, $B = C$ (Fig.). Alternative V will not arise, and we shall have

$$\begin{aligned} AA_2 &= AB_1 \cos \tfrac{1}{2}A = c \cos A \cos \tfrac{1}{2}A \\ &= 2R \cos A \cos^2 \tfrac{1}{2}A \\ &= R \cos A (1 + \cos A). \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$



Similarly,

$$\begin{aligned} A_{2n}A_{2n+2} &= R_{2n} \cos A_{2n}(1 + \cos A_{2n}) \\ &= -R(\cos 2^{2n}A - \cos^2 2^{2n}A)/2^{2n}. \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Hence

$$\begin{aligned} AA_{2n+2} &= R \cos A (1 + \cos A) \\ &\quad - R\sum_1^n (\cos 2^{2n}A - \cos^2 2^{2n}A)/2^{2n} \\ &= R(\tfrac{3}{2} + \cos A + S), \end{aligned}$$

where $$S=\sum_{1}^{r}\frac{(-)^{n-1}}{2^n}\cos 2^n A, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

and the series converges as $n\to\infty$.

This formula can be immediately checked in such simple cases as $A=\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{1}{4}\pi$, $\frac{3}{8}\pi$.

6. While there seems to be no available rule for summing the series in (12), either to n terms or to infinity, for a general value of A, all cases in which A is a *rational* fraction of π can be dealt with.

Even when the angles of the original triangle are all different, provided they are rational fractions of π, a stage is bound to come when, say, the nth pedal is equiangular with a previous one. If A, B are of the form $\pi p/q$, $\pi p'/q$, there will clearly be a repetition of the angles after not more than q^2 steps. In the case of the isosceles triangle, since B is then not independent of A, the repetition will come after not more than q steps; and the summation to be effected will be that of a convergent geometrical progression.

Tuckey (*l.c.*) shows that if the original angles are each a whole number of degrees, the order of the period is twelve, or a submultiple of twelve, and that the starting-point of the period comes not later than the third triangle.

7. A similar, and apparently even less tractable, problem arises when A_1, B_1, C_1 are the points in which the internal bisectors of the angles A, B, C meet the opposite sides. D. G. T.

CORRESPONDENCE.

MARK-FREQUENCIES.

To the Editor of the *Mathematical Gazette*.

SIR,—In a recent *Gazette* I found the suggestion—somewhat frivolous, if I may say so, for the pages of your austere periodical—that when marks are being analysed the way in which the various marks compete for the leading position might be regarded as a race and made the subject of a sweepstake.

This led me to study the way in which some marks for an algebra paper approached their final frequencies, with the following result.

The first eight marks to attain, and their order in attaining,

frequency 5 were	57, 78, 55, 56, 40, 42, 27, 35;
frequency 10 were	55, 75, 80, 78, 42, 81, 27, 56;
frequency 15 were	35, 42, 45, 55, 80, 48, 58, 59;

while the final order was

45 (fr. 23), 42 (fr. 20), 55 (fr. 18), 68 and 80 (fr. 17), 35, 39 and 58 (fr. 16).

The complete fading away of 57 after a good start, and the sudden spurt of 35 for the lead, followed by its collapse, are only less remarkable than that the ultimate winner did not appear among the leaders till more than half-way through the race.

All of which goes to show how unequal one school is to another and how misleading are our reports if they begin: "The standard in Algebra was in general very satisfactory." Yours, etc., C. O. T.

AN ISOPERIMETRIC THEOREM OBTAINED BY ELEMENTARY MEANS.

By A. D. Booth.

It is usually considered that the solution of isoperimetric problems is only to be achieved through the medium of the more or less advanced technique of variation calculus. The following theorem is derived using only the notions of pure geometry and elementary calculus, and is interesting since the method of proof is easily extended to several other problems of the same type.

"If a curve C, of fixed length, less than the minimum diameter of a convex curve Γ, is such that the area included between C and Γ is a maximum, the curve C is an arc of a circle orthogonal to Γ."

Before proving the theorem it is necessary to establish two particular results.

A. If two points A, B on a pair of straight lines meeting at a fixed angle in O are joined by a circular arc C, the area $AOBC$ is greatest when $OA = OB$.

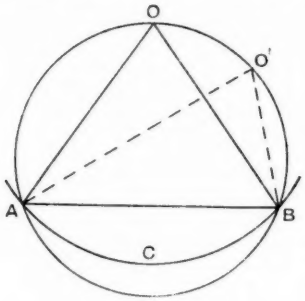


Fig. 1.

Referring to Fig. 1, since the lines meet at a constant angle the locus of O is a circle on AB as chord. The area $AOBC$ is equal to the sum of the areas ACB and AOB and, ACB being constant, $AOBC$ is greatest when AOB is a maximum, that is, when $OA = OB$.

B. Of all circular arcs ACB having constant length p, that making the area $AOBC$ a maximum is orthogonal to OA and OB.

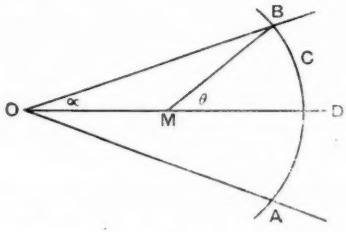


Fig. 2.

From A we see that it is necessary to consider only those arcs whose centres, M, lie on the bisector OD (Fig. 2) of the angle AOB.

Let
$$\text{angle } BOD = \alpha,$$
$$\text{,, } BMD = \theta,$$
$$\text{area } AOBC = A,$$
$$MB = r;$$

then
$$A = r^2(\theta - \tfrac{1}{2}\sin 2\theta) + r^2 \sin^2\theta \,.\, \cot\alpha,$$

and from the isoperimetric condition,
$$2r\theta = p;$$

whence, eliminating r,
$$A = \frac{p^2}{4}\left[\frac{1}{\theta} - \frac{1}{2\theta^2}\sin 2\theta + \frac{1}{\theta^2}\sin^2\theta \cot\alpha\right];$$

differentiating:
$$\frac{dA}{d\theta} = \frac{p^2}{4\theta^3}\left[-\theta - \theta\cos 2\theta + \sin 2\theta - 2\sin^2\theta\cot\alpha + \theta\sin 2\theta\cot\alpha\right]$$
$$= \frac{p^2}{2\theta^3\sin\alpha}\left[\sin\theta - \theta\cos\theta\right]\sin(\alpha - \theta).$$

At the turning point $\frac{dA}{d\theta} = 0$, whence $\theta = \alpha$ ($\theta = 0$ is not a solution), and since $\sin\theta \geqslant \theta\cos\theta$ when $0 \leqslant \theta \leqslant \frac{\pi}{2}$, we see that the value $\theta = \alpha$ gives a maximum. The centre of the arc making the area $AOBC$ a maximum therefore coincides with O and the arc is orthogonal to the two radial lines OA and OB.

It is convenient to prove the general theorem in two parts.

(1) That the isoperimetric curve making the included area a maximum is a circular arc.

(2) That this arc is orthogonal to the curve Γ.

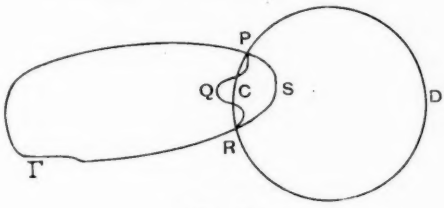


FIG. 3.

To prove (1) we recall the well-known result, that of all curves the circle has the greatest area for a given perimeter. In Fig. 3 assume, if possible, that the curve PQR (not circular) makes the area $PQRS$ a maximum. Let PCR be a circular arc of the same length; complete the circle $PCRD$. Then the curve $PQRD$ and the circle $PCRD$ have the same perimeter, and by the above theorem the circle has the greater area, that is, by subtraction of the common area $PSRD$, the area $PCRS$ is greater than the area $PQRS$; but

this is contrary to our assumption that the area $PQRS$ is a maximum, which is consequently false. Hence the curve PQR must be a circular arc.

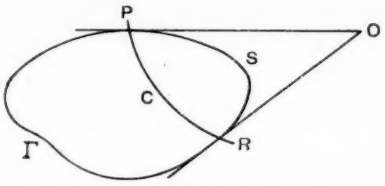


FIG. 4.

(2) In Fig. 4 let PCR be the circular arc making the area $PCRS$ a maximum, let PO and RO be the tangents to Γ at P and R. Now, since the tangents and the arcs of Γ coincide over an infinitesimal distance, we see from A that if $OP > OR$ (say) the whole arc PCR can be moved bodily with its extremities on PO, RO (and consequently on Γ over a small distance) to increase the area $OPCR$ and, by subtraction, the area $PCRS$. This is contrary to the assumption that the area $PCRS$ is a maximum, hence we must have $OP = OR$.

Next, suppose, in the case where $OP = OR$, that the arc PCR is not orthogonal to Γ (and consequently to OP and OR) at P and R, then by B we see that, by moving P and R along Γ and at the same time varying the radius of PCR, the area $OPCR$ can be increased. This is again contrary to the assumption that the area $PCRS$ is a maximum, hence the arc PCR must be orthogonal to Γ.

We have therefore proved the theorem enunciated.

A. D. B.

YORKSHIRE BRANCH.

REPORT FOR 1944-5.

THE number of Branch members is 58, of whom 25 are members of the Association.

Spring Meeting, 10th March, 1945, at Leeds University. Mr. C. W. Gilham, in a paper on "Prime Numbers and Waring's Problem", discussed the work of Euclid and Legendre concerning the number of primes, and of attempts by Greek mathematicians and Euler to express integers as sums of squares. This led to Waring's assertion, and its recent proof.

Summer Meeting, 16th June, at Leeds Girls' High School. Professor S. Brodetsky of Leeds University gave a lecture, illustrated by lantern slides, on "The Life and Work of Nicolas Copernicus". The significance of the work of this man, who, living in a world still under the influence of ancient prejudices, cleaned up our views on the solar system, was brilliantly shown.

Autumn Meeting, 13th October, 1945, at Leeds University. Colonel H. L. Hunter, M.B.E., M.A., Command Officer to the Northern Command, spoke on "The Education of the Soldier in the Modern Army", and told how A.B.C.A. was created to satisfy a felt need, of the unusual method used in the army to create thought and discussion, and the present work being done to prepare men for their return to civilian life. His talk was thoroughly enjoyed by all present, and aroused much discussion.

ALFRED LADLE, *Hon. Sec.*

ON CENTRIFUGAL FORCE.*

By E. G. Phillips.

What is centrifugal force? On selecting at random about a dozen elementary textbooks on Mechanics to find a definition, I was pleasantly surprised to discover (unless I have inadvertently missed it in any of them) that only four of them mention the name "centrifugal force", and two only of these four mention "centripetal force" as well. No doubt if I had had easy access to textbooks written for and used by engineers, the results would have been different; but as I am concerned only about mathematical students, I have not troubled to pursue my search except in the kind of textbooks likely to be used in schools which teach elementary mechanics as a branch of mathematics.

If my random selection of books gives a fair estimate of the position, it is most surprising to see how often the name "centrifugal force", and more often the vague uncertainties associated with it, occur in school examination scripts. They occur most commonly in certain types of problem, usually circular motion, and especially the conical pendulum.

Let us consider first the definitions given in the books of my random choice which are before me. For reference I shall call them *A*, *B*, *C* and *D*.

In *A*, the definition follows a discussion of the conical pendulum, which is said to be an example of *steady motion*. "It is indeed possible to ignore the effect of motion provided a force $m\omega^2 a$ acting on the particle in the direction of the outward radius is introduced to secure equilibrium against the horizontal component $T \sin \theta$. This *fictitious* (the italics are mine) force is called the *centrifugal force* due to the rotation." Then follows a statement that it is more in harmony with principles to say that the particle m, owing to its rotation, has an acceleration $\omega^2 a$ towards the centre. . . .

So here the definition, which is quite clearly expressed, is followed by an apology for its introduction, and the fictitious character of it is definitely mentioned.

In *B* we read: "If a man whirls in a circle a mass tied to one end of a string . . . the tension of the string exerts the necessary force on the mass to give it the required normal acceleration. . . .

"It *appears* to the man that the mass is trying to get away from his hand. For this reason a force, equal and opposite to the force necessary to give the particle its normal acceleration, has been called *its centrifugal force*. This is, however, a very misleading term. . . . A somewhat less misleading term is *centripetal force*."

In *C* we read: "It will be seen that if a particle is moving in a circle there is an apparent tendency in the particle to leave the circular path, necessitating a force directed towards the centre to keep it on the circumference. To this force there has been applied the somewhat misleading name of *centripetal force*.

"This force, whose magnitude is mv^2/r, must be due to the action of some

* [*Gazette*, V, p. 28. Report of a conference on the correlation of the teaching of mathematics and science.

Mr. W. J. Dobbs: ". . . Some time ago I was told by a young man of some scientific training, who sneered at the mathematician as a sort of glorified two-foot rule, that the forces which act on a small body whirling in a circle at the end of a string are five in number, viz. its momentum and its kinetic energy, both acting in the direction of motion; inertia, acting in the opposite direction; the centrifugal force outwards from the centre; and the tension of the string towards the centre."

This seems too good not to be resurrected. With Mr. Phillips' concurrence, I reproduce it here as a text for, or commentary on, his Note.—Ed.]

other body. And, since action and reaction are equal and opposite, the particle exerts on the latter body a force mv^2/r *away* from the centre, and this is called *centrifugal force*."

There is small wonder that ideas on this subject are so vague and confused in the minds of beginners if we examine critically the statements of B and C.

In D the term is only casually mentioned in discussing rotating liquids. Here a fictitious force $m\omega^2 r$ on each particle is introduced to transform the problem to a statical one, and it states: "There is no harm in speaking of this as a centrifugal force provided its artificial character be remembered."

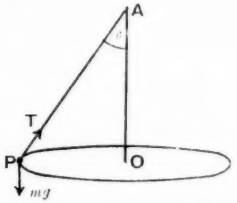
For students who have reached the state of maturity at which rotating liquids are usually discussed, there is no harm done, though the term "centrifugal force" could quite well be omitted altogether here.

In discussing d'Alembert's principle, A refers to the "reversed effective force" on a particle as a centrifugal force. Here again, at the stage reached by the student who has to consider d'Alembert's principle, the use of the term, though still quite unnecessary, could not do much harm.

If I had not seen so much evidence in examination scripts of the confusion of ideas resulting from the use of "centrifugal force" with beginners, I should not think it necessary to point out that it can all be avoided, and by a method which has the merit of being a correct application of the fundamental principles of dynamics. For example, in the case of the conical pendulum, where centrifugal force so often intrudes with such confusion, on writing down the equations of motion of the mass we get:

$$\text{Along } OP: \quad ma\omega^2 = T\sin\theta,$$
$$\text{parallel to } OA: \quad 0 = mg - T\cos\theta;$$

and nothing could be simpler.



Nevertheless, about half of the candidates in school examinations will put in a *force* at P, sometimes towards O, sometimes away from O, and the doubt in their minds is usually only too evident from the figures which they draw. What is the remedy? I am convinced that the only way to remove the uncertainty and confusion which at present exists is to take the drastic step of abolishing altogether, at any rate in elementary teaching, the term *centrifugal* (and, of course, *centripetal*) *force*. Then, and only then, will the vague and uncertain ideas associated with it also disappear.

Instead of debating whether it is a force "seeking the centre" or "fleeing from the centre", we should expedite its "flight" from the subject of mechanics altogether. E. G. P.

GLEANINGS FAR AND NEAR.

1477. Consider the first symphony of Brahms. . . . Its only postulates are a unison C and six drum-taps, the two coordinates of a great soaring curve that in its span encloses a grand edifice of sound.—*The Times*, **Dec. 29, 1944.**

ORNAMENTAL SQUARES AND TRIANGLES.

By T. R. Dawson.

Although it is difficult to think of a rational reason for being more interested in, say, the square number 698896 than in the square 700569, or for that matter in 4 or 49, there is apparently some psychological reason, and the pursuit of square numbers of particular digital forms has a long history. The publication of the attractive collection of digital curiosities in Note No. 1823 has suggested to me that there may be some interest in further figures from discoveries I have made in analogous fields. References in a few cases indicate previous publication in scattered problem columns.

Palindromic Squares.—Digital form $ABC \ldots CBA$. With an odd number of digits many algebraic squares are easily discovered, but the smallest example in an even number of digits is $836^2 = 698896$, a very old result. The next smallest case is $798644^2 = 637832238736$ (*Sphinx*, Mar., 1935).

Double Digit Squares.—$AABB \ldots$. The smallest is $88^2 = 7744$, often noticed in the tables and an incentive to find the next, exclusive of adding pairs of zeros. It turns out to be $74162^2 = 5500002244$ (*Sphinx*, Mar., 1935). This, like the preceding, was found by the well-known methods of "sifting" with quadratic residues, a sport which I imagine, though no angler, is the algebraist's equivalent for fishing.

Consecutive-Number Squares.—$N \,.\, N-1$, and $N \,.\, N+1$, the full-stops separating two numbers written side by side.

The smallest member of the descending family is $91^2 = 82\ 81$, and there are direct methods for finding them all. In this group I encountered my first "expanding" square. Compare

$$9901^2 = 9802\ 9801,$$

$$999001^2 = 998002\ 998001,$$

and we are led to examine :

$$9_m 0_{m-1} 1^2 = 9_{m-1} 8 0_{m-1} 2\ 9_{m-1} 8 0_{m-1} 1,$$

where the suffixes indicate the numbers of repetitions of the digits they affect (*L'Echiquier*, Oct., 1925).

The direct methods of solution permit limits to be assigned for the total number of squares of these classes.

Descending Squares ($2m$ digits) : If $10^m + 1$ is a product of r different prime factors, the number of squares lies between $2^{r-1} - 1$ and $2\,(2^{r-1} - 1)$. If "deficient" squares are admitted, *e.g.* $266327^2 = 070930070929$, where the $2m$ digits are made up by initial zeros, then the upper limit gives the exact number of all squares.

Ascending Squares ($2m$ digits) : If $10^{2m} + 1$ has r prime factors, the limits are r and $2r - 1$, the upper limit again giving the exact number if "deficient" squares, *e.g.* $2191^2 = 04800481$, are included.

Repeated-Number Squares.—NN. A very simple direct method seems to indicate that all the squares of this kind tend to a very great length. The shortest I have yet found is :

$$4^2 \,.\, 11^2 \,.\, 826446281^2 = 13223140496\ 13223140496.$$

Interpolated-Number Squares.—$N\,NM\,M$, that is, a number NM split into two parts and the same number put between them.

I gave a very lengthy example in the *Gazette*, Note No. 1481, in October, 1940. A simple direct method for obtaining these squares gives the only 12-digit specimens as : $406406^2 = 165\ 165836\ 836$, $593593^2 = 352\ 352649\ 649$,

and $998998^2 = 997\ 997004\ 004$, and a comparison of these with the only 8-digit specimens : $8989^2 = 80\ 8021\ 21$, and $9898^2 = 97\ 9704\ 04$, reveals another " expanding " item. In fact, in $9898 = 97\ 9704\ 04$, we can add two 9's to the root and two 9's and two 0's to the square as often as we please.

For squares of $8n$ digits we can derive from the foregoing another " expanding " square :

$$998999998999^2 = 998000\ 998000002001\ 002001.$$

Reversible Squares.—Pairs AB ... and ... BA.

The junior pair, $12^2 = 144$, $21^2 = 441$, naturally suggested this chase. Another small pair, $33^2 = 1089$, $99^2 = 9801$, is a particular case of a general theorem : in any scale of notation with radix $n^2 + 1$, the square of nn has digits 1, 0, $n^2 - 1$, n^2, and the square of n^2n^2 has digits n^2, $n^2 - 1$, 0, 1.

This class of squares is relatively rare, only 2 pairs in 3 digits, 1 pair in 4, 4 pairs in 5, and none in 6. The largest pair I have caught is : $3168^2 = 10036224$ and $6501^2 = 42263001$ (*Sphinx*, Sept., 1935).

Expanding Squares.—I have given three examples in earlier paragraphs, and must confess I fell quite by accident on a beautiful group as under. In all the following squares, we can continue to interpolate repetitions of the two middle digits indefinitely :

11102224,	1110022224, ...
11108889,	1110088889, ...
11115556,	1111155556, ...
11122225,	1112222225, ...
24990001,	2499900001, ...
44422225,	4442222225, ...
44435556,	4443355556, ...
44448889,	4444488889, ...
44462224,	4446622224, ...
99940009,	9994400009, ...
99960004,	9996600004, ...
99980001,	9998800001, ...
9999000025,	999900000025, ...

The third, fifth and eighth of these may start earlier at 1156, 249001, and 4489 respectively.

It is a somewhat daunting thought that when any of these or other categories of squares have all been found, we have only to transfer to any new radix to start the chase all over again.

Triangular Numbers.—When all the radixes are exhausted for all the squares, there will still remain fresh infinities of other " shapes " of number. The triangulars seem to have met with little attention hitherto.

Here are all the Palindromic Triangulars up to six digits : 1, 3, 6, 55, 66, 171, 595, 666, 3003, 5995, 8778, 15051, 66066, 617716 and 828828. Skipping a few, next come 178727871 in nine digits, and 1264114621 and 6172882716 in ten digits.

The last, compared with 617716 and 66, suggested a guess at the next member of a general group : 61728399382716, which proved to be correct and led to the wider generalisation that for any radix $2r$, we have the triangular numbers whose digits are :

$$(r+1)(r+1),$$

$$(r+1)1\,(r+2)(r+2)1\,(r+1),$$

.....

$$(r+1)1(r+2)2(r+3)\ldots m(r+m+1)(r+m+1)m\ldots(r+3)2(r+2)1(r+1),$$

e.g. in the scale of 12 we have the triangle :

718293*t*4*ee*4*t*392817 (*Sphinx*, March, 1937).

If the first triangular number is 1, then the nth triangular number terminating in n has only one solution up to 1000000, namely $\Delta 625 = 195625$, and if the " first " triangular number is taken as 0, the only result in the same range is $\Delta 128 = 8128$ (*Sphinx*, Sept., Oct., 1935).

The corresponding problem for squares was completely solved by P. Tédenat in 1814–15.

Consecutive-Number Triangles can be found by direct methods, and revealed two " expanding " results :

4950, 499500, 49995000, ...

5253, 502503, 50025003, ...

I cannot say how many analysts would add to these the " deficient " case :

0001, 000001, 00000001, ...

T. R. D.

1478. Soon after the Flood had subsided and the Ark grounded, Noah distributed the various pairs of inmates with instructions to settle down and multiply exceedingly. After a due interval, he made a tour of inspection to see if production was up to schedule, and found that output was well maintained, except in one case. When he arrived at the chosen settlement of the adders he saw that there were no additions, and inquired why. He was told that they could not very well multiply, because they were, after all, adders.

This was rather a poser for Noah, but the male adder said that he had thought of an idea which might work, if he could have a tree felled and the services of a carpenter. This was arranged.

Next time Noah went " progress chasing "—not such a new occupation as war-time executives think—he was delighted to see that the method had evidently proved successful, for there were swarms of young adders writhing about. Expressing his gratification, Noah asked how they had managed it. They explained that from the tree they had made some log tables, by the aid of which they were able to multiply by adding.—*The Motor*, October 11, 1944, p. 190.

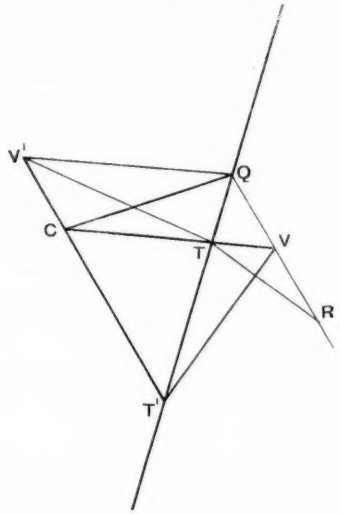
... I cannot see why they had to ask Noah for the loan of a carpenter to cut down trees, the latter surely in short supply after the world-wide flood. Why could they not have used the ship's log for their tables?—*The Motor*, October 25, 1944, p. 225. [Per Instr. Captain F. W. Shurlock.]

1479. Mathematics again is a subject in which we have much to learn on both sides. There are professors whose defence of mathematics reminds one of Socrates' defence of justice as being desirable in and for itself quite apart from any practical utility they may have for common life, or physical science, or navigating the sea. It was this school of thought which in the old days held up at least 90 per cent. of our school children by stubbornly defending the Euclidian ass's bridge-head. A study of the American method might still effect a great saving of time for prospective engineers. Our motto has been " the slower, the surer ". Americans insist that the learner should have, all the time, the feeling that he is getting forward. They prefer the short road, with plenty of revision should any symptom of unsoundness betray itself.—J. L. Paton, " Education as a Link between Great Britain and America ", *Journal of Education*, No. 863, June, 1941.

MATHEMATICAL NOTES.

1850. *Notes on conics.* 10 : *The invariants.*

1. Let the tangent at a point Q meet a pair of conjugate diameters p, p' in T, T', let QV be the ordinate to p, parallel to p', and let QV' be the ordinate to p', parallel to p. Then the radial measures ϖ, ϖ' of p, p' are the products $CV \,.\, CT$, $CV' \,.\, CT'$; the radial measure κ of the diameter CQ is CQ^2, and the radial measure κ' of the diameter conjugate to CQ, which is the diameter parallel to the tangent TT', is the product $TQ \,.\, QT'$.



2. Since $QT'/TT' = CV/CT$ and $TQ/TT' = CV'/CT'$, the elementary identity

$$CT^2 \,.\, QT' + CT'^2 \,.\, TQ + CQ^2 \,.\, T'T = -\, QT' \,.\, TQ \,.\, T'T$$

implies

$$CV \,.\, CT + CV' \,.\, CT' = CQ^2 + TQ \,.\, QT',$$

that is,

$$\varpi + \varpi' = \kappa + \kappa',$$

and since, with Q fixed, the pair of conjugate diameters p, p' is arbitrary,

The sum of the radial measures of a pair of conjugate diameters is a constant for the conic.

This constant is the *orthoptic constant*. To verify the property with which it is traditionally associated, we remark that, always,

$$CV' = VQ, \quad CT'/VQ = CT/VT,$$

implying

$$\varpi'/VQ^2 = VT \,.\, CT/VT^2.$$

If R is the point of contact of the second tangent from T, then V is the midpoint of QR, and therefore the tangents TQ, TR are at right angles

if and only if $VQ^2 = VT^2$, that is, if and only if $\varpi' = VT \,.\, CT$, that is, if and only if $\varpi + \varpi' = CT^2$.

3. Returning to the original figure we have, from the parallelisms, the areal equalities

$$\triangle CTY' = \triangle CTQ, \quad \triangle CVT' = \triangle CQT'.$$

Assigning directions of measurement to the several lines, we have

$$CT \,.\, CV' \sin \omega = TQ \,.\, CQ \sin \chi,$$

$$CV \,.\, CT' \sin \omega = CQ \,.\, QT' \sin \chi,$$

when ω is an angle from CT to CT' and χ an angle from CQ to TT'. Multiplying,

$$\varpi\varpi' \sin^2 \omega = \kappa\kappa' \sin^2 \chi.$$

If ϖ, ϖ' are the radial measures of a pair of conjugate diameters and ω is an angle between these diameters, the product $\varpi\varpi' \sin^2 \omega$ is a constant for the conic.

This constant may be called the *areal constant* of the conic. If the conic is an ellipse, the areal constant K is positive, and $4\sqrt{K}$ is the area of any circumscribed parallelogram whose sides are parallel to conjugate diameters. If the conic is a hyperbola, the areal constant is negative, one of the measures ϖ, ϖ' is positive and the other negative, and if ϖ is positive, there are no tangents parallel to p.

The figure, drawn for a hyperbola, emphasises that the sign-adjustments are automatic.

E. H. N.

1851. *Notes on conics.* 11 : *Gaskin's theorem.*

Let ABC be a triangle self-conjugate for a conic, let O be the centre of the conic, and let OA cut BC in X. Since A and X are conjugate points on the diameter OA, the radial measure of this diameter is $OA \,.\, OX$, and the power of X along the diameter is $OX^2 - OA \,.\, OX$, that is, $OX \,.\, AX$. Since X is the centre of the involution of conjugate points on the line BC and B and C compose a pair in that involution, the power of X along BC is $-XB \,.\, XC$, that is, $BX \,.\, XC$. But the ratio of the powers of X along the two lines BC, OA is the same as the ratio of the radial measures of the conic along the diameters parallel to these lines. Hence

The radial measure of the diameter parallel to BC is $BX \,.\, XC \,.\, OA/AX$.

This in effect is Gaskin's theorem. For if OA cuts the circumcircle of ABC again in R, the measure is $OA \,.\, XR$, and since the diameter parallel to BC is the diameter conjugate to OA, the sum of the radial measures along these two diameters, namely $OA \,.\, OX + OA \,.\, XR$, is the orthoptic constant :

The orthoptic constant of the conic is the power of the centre of the conic for the circumcircle of a self-conjugate triangle.

Unless the conic is an obtuse hyperbola, the result can be expressed in the traditional form, that the orthoptic circle of the conic is orthogonal to the circumcircle of the triangle, and it is from this form that the corresponding theorem for a parabola is deducible.

In a recent note in *The Mathematics Student* (xii, 105), Mr. M. Lakshmanamurti establishes Gaskin's theorem by a curious algebraical lemma, easily verified :

If

$$ax_2x_3 + by_2y_3 = ax_3x_1 + by_3y_1 = ax_1x_2 + by_1y_2 = 1,$$

then

$$\begin{vmatrix} ax_1^2+1 & x_1 & y_1 \\ ax_2^2+1 & x_2 & y_2 \\ ax_3^2+1 & x_3 & y_3 \end{vmatrix} = \begin{vmatrix} by_1^2+1 & x_1 & y_1 \\ by_2^2+1 & x_2 & y_2 \\ by_3^2+1 & x_3 & y_3 \end{vmatrix} = 0.$$

E. H. N.

1852. *Farey series.*

The Farey series $\mathfrak{F}_n$ of order n is the ascending series of irreducible fractions from 0 (written as 0/1) to 1 (written as 1/1) whose denominators do not exceed n. The series are of the utmost importance in the analytical theory of numbers. To derive $\mathfrak{F}_n$ from $\mathfrak{F}_{n-1}$ we have only to insert in their proper places the fractions whose denominator is n and whose numerators are the numbers less than and prime to n. The largest complete Farey series in print is constituted by the arguments in "a tabular series of complete decimal quotients for all the proper vulgar fractions of which, when in their lowest terms, neither the numerator nor the denominator is greater than 100", which is an appendix to Henry Goodwyn's *First Centenary of a Series of . . . Tables of . . . Quotients*", London, 1818. The same author designed a tabular series whose arguments would have constituted the Farey series of order 1000; the complete work was to have been in five parts, and the first part, a beautiful piece of printing, was published in 1823, but presumably support was not forthcoming, and no more appeared; the range of the published section is from 0/1 to 1/10 *.

The fundamental properties of Farey series can be expressed succinctly in the two theorems:

(1) *If* $bp-aq=1$, *then* a/b, p/q *are consecutive terms in the Farey series* $\mathfrak{F}_n$ *for all values of* n *from* $\max(b, q)$ *to* $b+q-1$;

(2) *If* a/b, p/q *are consecutive terms in any Farey series, then* $bp-aq=1$.

These familiar theorems are much simpler than is commonly thought. Th. 1 is nothing but the elementary theorem:

(3) *If* $bp-aq=1$ *and* h/k *is between* a/b *and* p/q, *then* $k \geqslant b+q$,

and Th. 2, as we shall see, follows directly from Th. 1. Perhaps this line of argument is avoided because Th. 3 is supposed to be hard to prove; if so, this is a sheer mistake. If $a/b < h/k < p/q$, then

$$bh-ak>0 \geqslant 1, \quad pk-qh>0 \geqslant 1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

and since the signs of inequality face the same way, we have

$$(bp-aq)h>a+p, \quad (bp-aq)k>b+q,$$

unless it is two equalities which hold in (a), in which case we have

$$(bp-aq)h=a+p, \quad (bp-aq)k=b+q.$$

This establishes Th. 3, with the addition:

(4) *In Th.* 3, *if* $k=b+q$, *then* $h=a+p$.

Suppose now that in a particular series $\mathfrak{F}_m$ the two fractions a/b, p/q are consecutive and do satisfy the condition $bp-aq=1$; let $h=a+p$, $k=b+q$. Identically,

$$bh-ak=1, \quad kp-hq=1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(b)$$

* A collaborator of Lehmer's (*Guide to Tables in the Theory of Numbers*, 1941) fails to notice that the 1823 title-page describes a plan and not an accomplishment.

and h is prime to k. Since the mediant h/k is necessarily between a/b and p/q, the last two fractions can not be consecutive in $\mathfrak{F}_k$, but by Th. 3 they are consecutive in $\mathfrak{F}_n$ for all values of n from m to $k-1$. In $\mathfrak{F}_k$, by Th. 4, the mediant is the only term between a/b and p/q. That is to say, in $\mathfrak{F}_k$ the interval from a/b to p/q is broken into precisely the two intervals $(a/b, h/k)$ and $(h/k, p/q)$, and by (*b*), the relation assumed between the ends of the original interval is reproduced between the ends of each of these two parts. It follows that, whatever interpolations take place between a/b and p/q as the order of the series is increased, consecutive terms in any $\mathfrak{F}_n$ for which $n > m$ satisfy the crucial condition if they lie in the original interval $(a/b, p/q)$. Since the condition is in fact satisfied by the two terms 0/1, 1/1 which constitute $\mathfrak{F}_1$, the general theorem is established.

As a corollary to Th. 2, we have a theorem, not often enunciated, which completes Th. 3 :

(5) *If* $bp - aq > 1$, *there is a fraction between* a/b *and* p/q *whose denominator is less than* $\max(b, q)$.

For if $m = \max(b, q)$, then a/b and p/q are terms in $\mathfrak{F}_m$, and since $bp - aq \neq 1$, they are not consecutive in $\mathfrak{F}_m$, and one of the terms between them is a neighbour of a term with denominator m and therefore has itself a denominator less than m.

An alternative deduction of Th. 2 from Th. 1 may seem even simpler than the deduction above. If h is prime to k, there is a unique integral solution $x = a$, $y = b$, of the equation $hy - kx = 1$ with the condition $0 < y < k$, and there is a unique integral solution $x = p$, $y = q$ of the equation $kx - hy = 1$ with the same condition. Since b and q are less than k, the fractions a/b and p/q belong to $\mathfrak{F}_k$; by Th. 1, since $hb - ka = 1$ there is no member of $\mathfrak{F}_k$ between a/b and h/k, and since $pk - qh = 1$ there is no member of $\mathfrak{F}_k$ between h/k and p/q. That is to say, the fundamental relation does hold between h/k and each of its neighbours in $\mathfrak{F}_k$. And if a fraction u/v for which $v < k$ is not a neighbour of h/k in $\mathfrak{F}_k$, the fractions can not be neighbours in any series whose order is greater than k.

The argument is sound, but surely to use it in the reverse direction is much more exciting. The theorem that, if h is prime to k, the equation $kx - hy = 1$ has a unique solution for which $0 < y < k$, is one of the fundamental theorems of arithmetic. There are three classical proofs : (i) by the theory of simple continued fractions, x/y or $(h-x)/(k-y)$ being the penultimate convergent of the fraction whose value is h/k ; (ii) by congruences, the remainders when the $k-1$ numbers $h, 2h, \ldots, (k-1)h$ are divided by k being shown to be all different ; (iii) by the principle of the minimum, the essence of the proof being that the non-zero numbers expressible in the form $xk - yh$ can not have a minimum greater than unity. We have now a fourth proof which is not in any sense a variant of one of these : x/y is the larger neighbour of h/k in the Farey series of order k, or y/x is the smaller neighbour of k/h in the Farey series of order h, according as k is greater or smaller than h. This is a valid proof, if Th. 2 is established independently. **E. H. N.**

1853. *Trigonometrical formulae.*

For $\tan \frac{1}{2}A$ in terms of a, b, c and s we can proceed thus :

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Hence

$$(b^2 + c^2 - a^2)(\sin^2 \tfrac{1}{2}A + \cos^2 \tfrac{1}{2}A) = 2bc\,(\cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A),$$

and so

$$\tan^2 \tfrac{1}{2}A = \{a^2 - (b-c)^2\}/\{(b+c)^2 - a^2\}$$
$$= (s-b)(s-c)/s(s-a).$$

If the formula for $\cos A$ in terms of $\tan \frac{1}{2}A$ is known and also the bookwork of ratios, the above proof may be considerably improved, but the most elementary form is given above.

As this formula is more generally used than the corresponding ones for $\sin \frac{1}{2}A$ and $\cos \frac{1}{2}A$, it seems a mistake to make it depend on them, as is done in so many textbooks. No originality is claimed for the proof here given, but it does not appear to be well known. J. J. WELCH.

1854. *An interesting identity.*

The following is a simple algebraic identity, but the simplest of the special cases was used by Gauss to produce interesting theorems in "Arithmetic".

Consider an infinite ordered sequence of elements

$$(p) \equiv p_1, p_2, p_3, \ldots,$$

and let q be a base number.

Let (n) denote $1 - q^{p_1+p_2+\ldots+p_n}$ and

$$(n)! \text{ denote } (1-q^{p_1})(1-q^{p_1+p_2})\ldots(1-q^{p_1+p_2+\ldots+p_n}).$$

Next consider a reversal of order in which $[n]!$ denotes

$$(1-q^{p_n})(1-q^{p_n+p_{n-1}})\ldots(1-q^{p_n+\ldots+p_1}).$$

Then very simply we obtain

$$1 - \frac{[n]}{(1)} + q^{p_1}\cdot\frac{[n]\,[n-1]}{(2)!} - q^{2p_1+p_2}\cdot\frac{[n]\,[n-1]\,[n-2]}{(3)!} + \ldots + (-)^n q^{\Delta_n}\cdot\frac{[n]!}{(n)!} \equiv 0;$$

the element numbers Δ are p_1, $2p_1+p_2$, $3p_1+2p_2+p_3$, $\ldots$. When the elements are units, the form reduces to a well-known elementary result in basic numbers.

F. H. JACKSON.

1855. *On Notes* 1802, 1803.

Mr. Birch obtains an approximation to the sum of the infinite series for $\log\{(1+z)/(1-z)\}$ by allowing the series to continue in geometric progression after the first two terms. The approximations he gives for $e^{1/n}$ and, in § 3, for $\log(1+x)$ can be obtained in the same way. I have often suggested to pupils that d'Alembert's ratio test, depending as it does on comparison with a convergent geometric progression, indicates this useful method of obtaining successive approximations, in the form of rational algebraical fractions, to the sums of the commoner power series. In particular, it gives a better approximation to $\sin x$ than that of Note 1803:

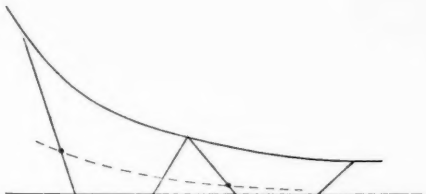
$$\begin{aligned}\sin x &= x - x^3/6 + x^5/120 - \ldots \\ &= x - \tfrac{1}{6}x^3(1+\tfrac{1}{20}x^2)^{-1} \text{ approximately} \\ &= -7x/3 + 200x/3(20+x^2),\end{aligned}$$

with an error term $11x^3/50400$.

An interesting point is that the method works even in the critical cases in which d'Alembert's test fails. Successive partial sums of the series for $\log(1+x)$, when $x=1$, form a very slowly convergent sequence. The sum of the first ten terms is 0·6456 ..., but if we assume the series to continue in geometric progression with common ratio $-9/10$, given by the last two terms, the sum is 0·6930 ... (and $\log 2 = 0{\cdot}69315\ldots$).

A well-known device in this series and in Gregory's series for $\pi/4$ is to halve the last term in the partial sum, a process which cannot be justified by comparison with a geometric progression, since it amounts to setting the common ratio equal to 1. The easiest explanation to a class is afforded by a graph of

the successive partial sums and of the adjusted sums, which are represented by the middle points of straight line portions.



Now, since the sequence of adjusted sums has the same property, namely that its odd members form an upper decreasing sequence and its even members a lower increasing sequence, it is useful to repeat the halving process; in fact, starting with a finite number of terms of the series we finally adjust the sum to a single number which may serve as the required approximation. But we can still improve the result. Using six terms of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ the partial sums and the successive sequences of middle values are tabulated in columns :

1	·75	·708333	·697917	·694792	·693750 (*c*)
·5	·666667	·6875	·691667	·692708 (*d*)	
·833333	·708333	·695833	·693750		
·583333	·683333	·691667			
·783333	·7 (*b*)				
·616667 (*a*)					

(*a*) is the sum of six terms.

(*b*) is the sum obtained by halving the sixth term—a great improvement.

(*c*) is the result of the complete halving process and is better still.

But obviously (*c*) is too large, since it is the first term of an upper decreasing sequence, while (*d*) is too small, belonging to a lower sequence. Using the mean of (*c*) and (*d*) we have $\log 2 = \cdot 69323\ldots$.

If we apply this method to four terms of the series $4(1-\frac{1}{3}+\frac{1}{5}-\ldots)$ we obtain $3\frac{1}{7}$ as an approximate value of π. C. G. PARADINE.

1856. *Quelques nombres curieux.*

Dans une brochure qui vient de paraître,* nous avons consacré un chapitre à des nombres curieux obtenus avec les chiffres 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, pris une fois, sujet qui a été considéré avant nous par H. E. Dudeney.† Voici encore quelques nombres curieux à y ajouter.

* *Les Récréations mathématiques* (Parmi les nombres curieux), *Mathesis* (Supplément), t. LIV.

† *The Canterbury Puzzles.*

Problèmes. 1. *Avec les chiffres* 1, 2, 3, 4, 5, 6, 7, 8, 9 *pris une fois, former trois nombres* N, n, r *tels que* n *et* r *soient la racine cubique à l'unité près par excès de* N *et le reste correspondant.*

Par hypothèse, on doit avoir successivement

$$N = n^3 - r, \text{ avec } 0 \leqslant r \leqslant 3n(n+1),$$

$$N + n + r = M(9) + (1+2+3+4+5+6+7+8+9) = M(9),$$

de sorte que

$$n^3 - r + n + r = n(n^2+1) = M(9),$$

ce qui exige $n = M(9)$, car n^2+1 est premier avec 9. De plus, on a $18 \leqslant n \leqslant 81$, et comme les deux chiffres de n et les quatre chiffres à gauche de N doivent être différents, ce qui écarte $n = 54$, 63, 72 et 81, N n'a que cinq ou quatre chiffres, suivant que $n = 27$, 36, 45 ou que $n = 18$.

Si N a cinq chiffres et si $n = 27$, $n^3 = 19683$, $13 \leqslant r \leqslant 98$. La seule soustraction à envisager

$$\begin{array}{r} 19683 \\ -\quad ab \\ \hline 196cd \end{array}$$

montre déjà que les trois chiffres à gauche de N sont 1, 9, 6. Les autres chiffres a, b, c, d sont 3, 4, 5, 8 dans un certain ordre et $ab + cd = 83$; d'où $a = 4$ ou 3, $b = 5$ ou 8, $c = 3$ ou 4, $d = 8$ ou 5. On a donc les solutions *primitives*

$$N = 19638 = 27^3 - 45 \quad \text{et} \quad N = 19645 = 27^3 - 38$$

desquelles dérivent

$$N = 19635 = 27^3 - 48, \quad N = 19648 = 27^3 - 35.$$

Les hypothèses où $n = 36$, $n = 45$, traitées de la même façon ne donnent pas de solution pas plus que celle où $n = 18$.

N.B.—Il n'y a pas de nombres N, n, r formés de dix chiffres tous différents jouissant de la même propriété.

2. *Avec les chiffres* 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *pris une fois, former trois nombres* N, n, r, *tels que* n *et* r *soient la racine quatrième à l'unité près par excès de* N *et le reste correspondant.*

Dans ce problème

$$N = n^4 - r, \quad N + n + r = M(9) + (0+1+2+\ldots+9) = M(9),$$

d'où

$$n^4 - r + n + r = n(n^3+1) = n(n+1)(n^2-n+1) = M(9),$$

ce qui suppose $n = 9$, car n est premier avec n^3+1, ou $n+1 = M(9)$, ou $n^2-n+1 = M(9)$, ou enfin $n+1 = M(3)$ et $n^2-n+1 = M(3)$, car $n+1$ et n^2-n+1 ont 3 pour p.g.c.d.

En procédant comme au problème précédent, on obtient les solutions primitives

$$N = 37596 = 14^4 - 820, \qquad N = 37820 = 14^4 - 596,$$

$$N = 83062 = 17^4 - 459, \qquad N = 83459 = 17^4 - 062,$$

$$N = 1048567 = 32^4 - 9.$$

3. *Avec les chiffres* 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *pris une fois, former trois nombres* N, n, r *tels que* n *et* r *soient la racine sixième à l'unité près par excès de* N *et le reste correspondant.*

$$N = n^6 - r, \quad N + n + r = M(9) + (0+1+2+\ldots+9 = M(9),$$

d'où

$$n^6 - r + n + r = n(n^5 + 1) = n(n+1)(n^4 - n^3 + n^2 - n + 1) = M(9),$$

ce qui exige $n = M(9)$, ou $n + 1 = M(9)$, c'est-à-dire $n = 9$ ou $n = 8$, ca $n^4 - n^3 + n^2 - n + 1$ est premier avec 3. Ces hypothèses donnent les solutions

$$N = 530827 = 9^6 - 614, \qquad N = 530614 = 9^6 - 827,$$
$$N = 261739 = 8^6 - 405, \qquad N = 261405 = 8^6 - 739.$$

N.B.—Avec les chiffres 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, pris une fois, on ne peut former trois nombres N, n, r tels que n et r soient racine cinquième, septième, huitième ou neuvième de N et le reste correspondant. V. THÉBAULT.

1857. *Carré inscrit dans un quadrant.*

1. Une construction du carré $CDEF$ inscrit dans un quadrant, de rayon $OA = OB = R$, les sommets E, F étant sur l'arc, a été donnée par Sanjana (*Educational Times*, 1897, p. 47, Reprints) et par (M.), (*Journal de Mathématiques Elémentaires de G. de Longchamps*, 1897, p. 108).

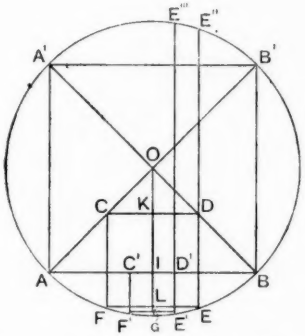
2. En voici une autre avec celle du carré $C'D'E'F'$ inscrit dans le segment sous-tendu par la corde AB du quadrant, les sommets E', F' étant aussi sur l'arc. Désignant x, y les longueurs des côtés des carrés $CDEF$, $C'D'E'F'$, K, I, L, G les intersections de la bissectrice de l'angle AOB avec CD, AB, EF, $E'F'$, on a

$$OF^2 = OL^2 + LF^2 = (OK + x)^2 + \tfrac{1}{4}x^2 = 5x^2/2 = R^2 ; \quad \text{d'où } x = R\sqrt{(\tfrac{2}{5})} ;$$
$$OF'^2 = OG^2 + GF'^2 = (OI + y)^2 + \tfrac{1}{4}y^2 = (\tfrac{1}{2}R\sqrt{2} + y)^2 + \tfrac{1}{4}y^2 = R^2 ;$$

d'où l'équation

$$5y^2 + 4Ry\sqrt{2} - 2R^2 = 0,$$

dont les racines $y = \frac{1}{5}R\sqrt{2}$, $y_1 = -R\sqrt{2}$, sont les côtés des carrés $C'D'E'F'$ et $ABA'B'$ inscrits dans les segments déterminés par le côté AB du carré inscrit dans le cercle (O, R). De ces formules, il résulte que $y = C'D' = AB/5$. Il suffit donc de partager AB en cinq parties égales pour obtenir les sommets C', D' et, par suite, le carré $C'D'E'F'$.



De plus, *la corde BE' a même longueur que le côté du carré $CDEF$, car*

$$BE'^2 = BD'^2 + D'E'^2 = (2AB/5)^2 + (AB/5)^2 = 2R^2/5 = x^2.$$

Dès lors, pour construire le carré $CDEF$, il suffit de tracer les parallèles à OG distantes de cette droite d'une longueur égale à $\frac{1}{2}BE'$, ou bien de mener, au-dessous de AB, la tangente parallèle à AB au cercle de centre O tangent à la corde BE'.

3. La figure présente d'autres curiosités faciles à vérifier.

(*a*) Puisque $AB/CD = CD/C'D' = \sqrt{5}$, *l'aire du carré* $CDEF$ *est moyenne proportionnelle entre celles des carrés* $ABA'B'$ *et* $C'D'E'F'$.

(*b*) *Si* ED *et* $E'D'$ *rencontrent respectivement la circonférence* (O, R) *en* E'' *et* E''', *on a* $EE'' = 3ED$, $E'E''' = 7E'D'$.

(*c*) Les côtés des carrés inscrits dans les deux segments formés par un côté de l'hexagone régulier inscrit dans le cercle (O, R) ayant pour longueurs

$$z = \tfrac{1}{5}R(\sqrt{17} - 2\sqrt{3}), \quad t = \tfrac{1}{5}R(\sqrt{17} + 2\sqrt{3}),$$

la demi-diagonale du carré $CDEF$ *leur est moyenne proportionnelle.*

V. THÉBAULT.

1858. *A propos des points de Feuerbach d'un triangle.*

Le cercle des neuf points (O_9) d'un triangle ABC dans lequel $BC = a$, $CA = b$, $AB = c$, est tangent aux cercles inscrit (I) et exinscrits (I_a), (I_b), (I_c) de centres I, I_a, I_b, I_c, en des points ϕ, ϕ_a, ϕ_b, ϕ_c (points de Feuerbach). Soient O le centre du cercle circonscrit, de rayon R, et H l'orthocentre du triangle ABC. Si l'on applique le théorème de Stewart au triangle OO_9I_a, de l'égalité

$$O_9I_a \cdot OI_a^2 + \phi_aI_a \cdot OO_9^2 = O_9I_a \cdot O\phi_a^2 + O_9O_a \cdot \phi_aI_a \cdot O_9I_a$$

il résulte que

$$O\phi_a^2 = R^2 + (OI_a^2 - R^2)(OH^2 - OI_a^2)/2OI_a^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

et l'on obtient des formules analogues pour $O\phi_b^2$, $O\phi_c^2$. De son côté, le théorème de la médiane appliqué au triangle ϕ_aOH donne

$$H\phi_a^2 = \tfrac{1}{2}(OH^2 + R^2) - O\phi_a^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

et il y a des relations similaires pour $H\phi_b^2$, $H\phi_c^2$.

Dès lors, si $OH = OI_a$, il vient, à la fois, $O\phi_a = R$, $H = \rho$, ρ étant le rayon du cercle conjugué au triangle fondamental, et réciproquement, car $OI_a > R$.

Théorème. *Dans un triangle, la condition nécessaire et suffisante pour que l'un des points de Feuerbach* ϕ_a, ϕ_b, ϕ_c *soit sur le cercle circonscrit est que le centre de ce cercle soit équidistant de l'orthocentre et du centre du cercle exinscrit sur lequel le point* ϕ_i $(i = a, b, c)$ *est situé. Ce point de Feuerbach est en outre sur le cercle conjugué.*

Pour construire ce triangle spécial ABC étant donnés les cercles (O) et (I_a), de rayon r_a, on détermine d'abord le lieu de H. Comme

$$OH^2 = 2\rho^2 + R^2, \quad I_aH^2 = \rho^2 + 2r_a^2,$$

on a la relation

$$2HI_a^2 - OH^2 = 4r_a^2 - R^2,$$

qui définit un cercle (ω) ayant pour centre le symétrique de O, par rapport à I_a, et pour rayon $R + 2r_a$. Le cercle de centre O et de rayon OI_a coupe (ω) en H_1 et H_2. Les triangles cherchés ont pour côtés les tangentes communes au cercle (ω) et à la conique enveloppe des côtés des triangles inscrits à (O) et d'orthocentres H_1 ou H_2. La construction n'est pas possible par la règle et le compas.

Ces remarques nous ramènent au triangle particulier ABC dans lequel les angles $A = 4\pi/7$, C ou $B = 2\pi/7$, B ou $C = \pi/7$ sont les termes d'une progression géométrique de raison 2 dont nous avons donné de nombreuses propriétés *

* *Mathesis*, 1913, pp. 204–8 ; 1938, pp. 169–70.

auxquelles il faut ajouter que *le point de Feuerbach* ϕ_b (ou ϕ_c) *coïncide avec l'une des intersections des cercles* (O), $(H_1\rho)$, (O_9) et que *le sine-triple-angle-circle* * *se réduit à une droite parallèle à la symétrique de* CB, *par rapport à* BA (*ou* CA).

Corollaire. *Dans un triangle, pour que deux points de Feuerbach soient sur le cercle circonscrit, il faut et il suffit que le cercle circonscrit au triangle ayant pour sommets l'orthocentre et les centres des cercles exinscrits sur lesquels les points de Feuerbach sont situés, soit concentrique au cercle circonscrit.*

Ce triangle ABC est isocèle, car si $OH = OI_b = OI_c$,

$$AB = AC = AI_b = AI_c = \sqrt{(2Rr_b)}. \quad \text{De plus,}$$

$$OH^2 = R^2(1 + 8\cos 2B \cos B) = R^2 + 2Rr_b,$$

$\sin^2 B = r_b/2R$, et finalement on a l'équation

$$2r_b^2 - 7Rr_b + 4R^2 = 0,$$

dont la plus petite racine $r_b = \frac{1}{4}R(7 - \sqrt{17}) = r_c$ convient seule et donne

$$b^2 = c^2 = \tfrac{1}{2}R^2(7 - \sqrt{17}).$$

Dans ce triangle spécial, les points de Feuerbach ϕ_b, ϕ_c coïncident avec les deux intersections des cercles (O), (H,ρ), (O_9). V. THÉBAULT.

1859. *Un théorème sur le point de Feuerbach.*

Le théorème signalé par M. H. V. Gulasekharam (Note 1506, XXV, p. 55 ; Note 1685, XXVII, p. 207) est bien connu.† Il constitue d'ailleurs un cas particulier d'une proposition générale qu'il est bon de savoir.

Considérons un triangle ABC, une conique circonscrite (S) et un point quelconque P de cette conique. Soient A', B', C' les points où les droites AP, BP, CP rencontrent les côtés BC, CA, AB du triangle. Les points A', B', C' sont les faux sommets et le point de rencontre des diagonales du quadrilatère complet qui a pour sommets A, B, C, P. Chacun des sommets du triangle $A'B'C'$ a pour polaire par rapport à (S) le côté opposé. Le triangle $A'B'C'$ est donc conjugué par rapport à (S). Il s'ensuit, d'après le théorème de Faure : *le cercle circonscrit à un triangle conjugué par rapport à une conique* (S) *est orthogonal à son cercle orthoptique*, que le cercle circonscrit au triangle $A'B'C'$ est orthogonal au cercle orthoptique de (S).

Si (S) est une hyperbole équilatère, le cercle orthoptique se réduit à son centre et le cercle $A'B'C'$ passe par son centre. Donc, *le cercle circonscrit à un triangle conjugué par rapport à une hyperbole équilatère passe par le centre de cette hyperbole.* Or, le centre $I \equiv P$ de l'un des cercles tritangents du triangle ABC appartenant ainsi que les sommets A, B, C à l'hyperbole de Feuerbach, le triangle formé par les pieds A', B', C' des bissectrices correspondantes est conjugué par rapport à cette hyperbole et le cercle $A'B'C'$ passe par le centre de l'hyperbole, c'est-à-dire par le point de Feuerbach correspondant.

N.B.—Il en est de même du cercle qui passe par les points de contact du cercle (I) sur les côtés du triangle ABC. V. THÉBAULT.

1860. *Proof that every angle is a multiple of two right-angles.*

It is obvious that the necessary and sufficient condition that the equation

$$\tan(A + B) = \tan B$$

* Tucker, *Educational Times*, Dec., 1886.

† Meytelon, *Journal de Vuibert*, t. 38, p. 92 ; V. Thébault, *Annales de la Soc. Scient. de Bruxelles*, t. 54, série A, p. 153.

should hold is that $A = n\pi$. But suppose that $\tan B = i$, then

$$\tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$= \frac{\tan A + i}{1 - i \tan A}$$
$$= i = \tan B.$$

Hence $A = n\pi$ for all A.

The connection with the idea of isotropic lines is evident. It is also interesting to note that for any non-constant function which is obtained from the exponential function by a bilinear transformation there are two and only two values which the function cannot take. Thus if

$$f(z) = \frac{ae^z + b}{ce^z + d},$$

the equation $f(z) = k$ has no solution if $k = a/c$ or b/d. A. G. D. WATSON.

1861. *On Note* 1815.

The proof given is interesting; but the result—that there are in general three pedal lines passing through a given point P—follows at once from the fact that the various pedal lines are the tangents to a three-cusped hypocycloid, tangential to and concentric with the nine-point circle of the triangle. And this is fundamental; as it readily appears from a figure that for points inside the hypocycloid the three tangents are all real, while for points outside the curve two are imaginary. PERCY J. HEAWOOD.

1862. *Co-normal circles of a parabola.*

1. If the three normals are drawn from a point to a parabola, it is easy to prove that the circle through their feet passes through the vertex. But to find the equation of this circle is not so simple, and the following "trick" solution may be found acceptable. It was shown to the writer at Cambridge many years ago.

The condition that the normal at (x, y) to $y^2 = 4ax$ should pass through (α, β) is

$$2a(\beta - y) + y(\alpha - x) = 0.$$

Multiply throughout by y, put $y^2 = 4ax$ and divide by $2a$; then

$$y(\beta - y) + 2x(\alpha - x) = 0;$$

i.e. the feet of the normals from (α, β) lie on

$$2x^2 + y^2 - 2\alpha x - \beta y = 0,$$

and also on

$$(2x^2 + y^2 - 2\alpha x - \beta y) + (y^2 - 4ax) = 0,$$

i.e.
$$2(x^2 + y^2) - 2x(2a + \alpha) - \beta y = 0,$$

or
$$x(x - 2a - \alpha) + y(y - \tfrac{1}{2}\beta) = 0. \quad \text{.....(i)}$$

Let us call this the co-normal circle of (α, β). The power of (α, β) with respect to this circle is

$$-2a\alpha + \tfrac{1}{2}\beta^2 \quad \text{or} \quad \tfrac{1}{2}(\beta^2 - 4a\alpha).$$

Hence if the ordinate through a point O cut a parabola in P and P' and the co-normal circle of O in Q and Q', then

$$OQ \,.\, OQ' = \tfrac{1}{2} OP \,.\, OP'.$$

In general, let

$$f(x, y) \equiv (lx + my)^2 + 2gx + 2fy + c$$
$$\equiv (lx + my + \lambda)^2 - 2\mu(mx - ly + \nu) = 0,$$

where $\lambda = (gl + fm)/(l^2 + m^2)$, $\mu = (fl - gm)/(l^2 + m^2)$, $\nu = (\lambda^2 - c)/2\mu$.

By (i) the equation of the co-normal circle of (α, β) is

$$(lx + my + \lambda)\{l(x - \tfrac{1}{2}\alpha) + m(y - \tfrac{1}{2}\beta) + \tfrac{1}{2}\lambda\}$$
$$+ (mx - ly + \nu)\{m(x - \alpha) - l(y - \beta) - \mu\} = 0,$$

while the power of (α, β) with respect to this is

$$\{\tfrac{1}{2}(l\alpha + m\beta + \lambda)^2 - \mu(m\alpha - l\beta + \nu)\}/(l^2 + m^2)$$

or

$$f(\alpha, \beta)/2(l^2 + m^2).$$

2. The condition that the normal at (x, y) to $b^2x^2 + a^2y^2 - a^2b^2 = 0$ should pass through (α, β) is

$$a^2(\alpha - x)/x = b^2(\beta - y)/y\ ;$$

i.e. the feet of the normals from (α, β) lie on

$$(a^2 - b^2)xy + b^2\beta x - a^2\alpha y = 0.$$

Thus the parabolas through these feet are

$$b^2x^2 + a^2y^2 - a^2b^2 \pm 2ab\{(a^2 - b^2)xy + b^2\beta x - a^2\alpha y\}/(a^2 - b^2) = 0$$

or

$$\phi(x, y) \equiv (bx \pm ay)^2 \pm \frac{2ab}{a^2 - b^2}(b^2\beta x - a^2\alpha y) - a^2b^2 = 0.$$

We have

$$\phi(\alpha, \beta) = (b\alpha \pm a\beta)^2 \mp 2ab\alpha\beta - a^2b^2$$
$$= b^2\alpha^2 + a^2\beta^2 - a^2b^2.$$

Hence the power of (α, β) with respect to the co-normal circles of (α, β) of these parabolas, viz. $\phi(\alpha, \beta)/2(a^2 + b^2)$, is the same for both. We thus have the following theorem:

If the two parabolas are drawn through the feet of the normals from a point to an ellipse, then the radical axis of the co-normal circles of the point passes through it.

N. M. GIBBINS.

1863. *When is a line not a line?*

The equation

$$\begin{vmatrix} 2x + 3y - 4 & 3x + 7y - 2 \\ 2x + 3y + 1 & 3x + 7y - 5 \end{vmatrix} = 0$$

is obviously the equation of a straight line. In fact it is the line-at-infinity, because it passes through the point common to

$$2x + 3y - 4 = 0 \quad \text{and} \quad 2x + 3y + 1 = 0$$

and through another point distinct from this but of the same sort. But it is not the line-at-infinity, because it contains the point (0, 0·5) and other easily accessible points. Well?

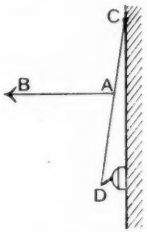
NORMAN ANNING.

1864. *A distressing problem successfully solved.*

Ever since I was a schoolboy nightly threatened with dire penalties for reading in bed, I seem to have been dogged through life by a series of bedrooms without bedside lamps. In those bedrooms in which my stay has been other than fleeting I have usually rigged up the normal string from electric switch

to bedside *via* a suitable hook, nail or "what have you". For over twenty years I have been continually surprised at the heavy tug that has been necessary to operate the switch, a tug that has seemed grossly in excess of anything that the normal laws of friction could add to the dainty touch usually sufficient to operate the switch. No doubt many readers of this note will have experienced the same sickening sensation of pulling the string from a well-snuggled-down position in bed on a frosty night and feeling it snap while the light goes on shining, unblinkingly and sardonically as one scrambles out into the chill to operate the switch manually.

The solution to the problem came to me recently, here in exile in India, where the situation is complicated by mosquito nets. I suddenly realised how the triangle of forces could be combined with the Indian Rope Trick to turn a small tension into a large one at right angles to it and avoid once and for all the disappointments caused by the snapping of over-frayed string.



AB string to reader-in-bed; *CAD*, string from fixed point *C* to switch *D*.

The diagram shows clearly how it is done. Obviously the ambitious can have a second string to their bed and put the light on in a similar fashion.

From my charpoy and mosquito net in the tropics I send this really useful example of applied mathematics to help those members of the Mathematical Association who had not thought of it for themselves. Of course you need the electric light!

SIDNEY F. TRUSTRAM.

1865. *On the calculation of logarithms.*

Euler in his *Elements of Algebra* gives the following simple method for calculating logarithms. He takes $\log_{10} 2$ as an example.

Let $\log_{10} 2 = x$, so that $10^x = 2$.

$$\begin{aligned}
10^0 &= 1, \quad 10^1 = 10; \quad \text{hence} \quad 0 < x < 1.\\
(10^{1/2})^2 &= 10, \quad 2^2 = 4; \quad \text{hence} \quad 0 < x < \tfrac{1}{2}.\\
(10^{1/3})^3 &= 10, \quad 2^3 = 8; \quad \text{hence} \quad 0 < x < \tfrac{1}{3}.\\
(10^{1/4})^4 &= 10, \quad 2^4 = 16; \quad \text{hence} \quad \tfrac{1}{4} < x < \tfrac{1}{3}.
\end{aligned}$$

Using, though he does not say so, the result that $(a+c)/(b+d)$ lies between a/b and c/d, he proceeds:

$$\begin{aligned}
(10^{2/7})^7 &= 100, \quad 2^7 = 128; \quad \text{hence} \quad \tfrac{2}{7} < x < \tfrac{1}{3}.\\
(10^{3/10})^{10} &= 1000, \quad 2^{10} = 1024; \quad \text{hence} \quad \tfrac{3}{10} < x < \tfrac{1}{3},
\end{aligned}$$

and so on, to any required degree of accuracy.

H. F. SANDHAM.

1866. *Factorisation.*

In the *Report of the Conference on School Certificate Mathematics*, p. 4, the factorisation of $a^2b+b-ab^2-a$ is categorised as "awkward". I suggest that the following approach to factorising will remove the stigma.

(*a*) The first rule is "take out any factor common to each term".

(*b*) To factorise a four-term expression, "take out the terms in pairs and apply the first rule to each pair". It may be necessary to group in pairs in more than one way before a common factor of the grouped pairs appears. Then apply the first rule again.

Thus
$$\begin{aligned} & a^2b+b-ab^2-a \\ &= (a^2b+b)-(ab^2+a) \\ &= b(a^2+1)-a(b^2+1)\text{—a failure.} \end{aligned}$$

A second grouping gives
$$\begin{aligned} & a^2b-ab^2+(b-a) \\ &= ab(a-b)+(b-a) \\ &= (a-b)(ab-1). \end{aligned}$$

I. FITZROY JONES.

1867. *On Note* 1771. *The Iso triangle.*

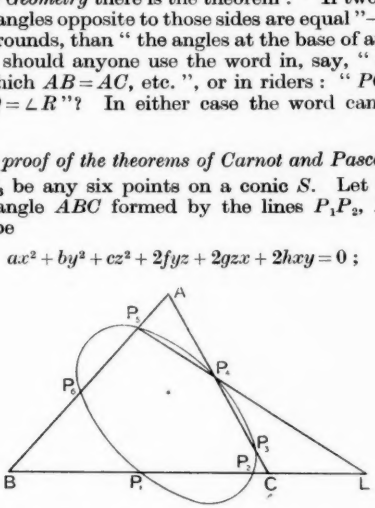
Granted that "they will not learn how to spell *isosceles*", we can either adopt an abbreviation, as Mr. Tuckey suggests, or stop using the word altogether. We are, in fact, well on the way to the latter expedient. In Durell's *Elementary Geometry* there is the theorem: "If two sides of a triangle are equal, then the angles opposite to those sides are equal"—surely very much better, on several grounds, than "the angles at the base of an isosceles triangle are equal". Why should anyone use the word in, say, "Given an isosceles triangle ABC in which $AB=AC$, etc.", or in riders: "$PQ=PR$, $\therefore$ $\triangle PQR$ is isosceles, $\therefore$ $\angle Q=\angle R$"? In either case the word can be omitted with advantage.

G. A. GARREAU.

1868. *Analytical proof of the theorems of Carnot and Pascal.*

Let $P_1, P_2, \ldots P_6$ be any six points on a conic S. Let the equation of S referred to the triangle ABC formed by the lines P_1P_2, P_3P_4, P_5P_6, using areal coordinates, be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0;$$



let the points $P_1, P_2, \ldots P_6$ be respectively

$$(0, y_1, z_1), (0, y_2, z_2), (x_3\ 0, z_3), (x_4, 0, z_4), (x_5, y_5, 0), (x_6, y_6, 0).$$

Then P_1, P_2 are given by

$$by^2+cz^2+2fyz=0.$$

Hence $$y_1y_2/z_1z_2 = c/b,$$

and similarly $$z_3z_4/x_3x_4 = a/c, \quad x_5x_6/y_5y_6 = b/a.$$

Thus $$(y_1y_2/z_1z_2) \cdot (z_3z_4/x_3x_4) \cdot (x_5x_6/y_5y_6) = 1. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Since $y_1/z_1 = BP_1/P_1C$, etc., this gives Carnot's theorem :

$$(BP_1 \cdot BP_2/CP_1 \cdot CP_2) \cdot (CP_3 \cdot CP_4/AP_3 \cdot AP_4) \cdot (AP_5 \cdot AP_6/BP_5 \cdot BP_6) = 1.$$

The equation of the line P_4P_5 is

$$\begin{vmatrix} x, & y, & z \\ x_4, & 0, & z_4 \\ x_5, & y_5, & 0 \end{vmatrix} = 0,$$

so that if P_4P_5 meets BC at L the coordinates of L are readily found to be $(0, x_4y_5, -z_4x_5)$.

Similarly, P_6P_1 meets P_3P_4 at $M(-x_6y_1, 0, y_6z_1)$ and P_2P_3 meets P_5P_6 at $N(z_2x_3, -y_2z_3, 0)$.

Now equation (i) can be written :

$$\begin{vmatrix} 0, & x_4y_5, & -z_4x_5 \\ -x_6y_1, & 0, & y_6z_1 \\ z_2x_3, & -y_2z_3, & 0 \end{vmatrix} = 0,$$

and so L, M, N are collinear, which is Pascal's theorem. G. A. GARREAU.

1869. *The difference of two squares.*

I am indebted to a pupil for the following ingenious misuse of the difference of two squares, which has the doubtful advantage of producing the right answer :

$$\begin{aligned} &\sin^2(\alpha+\theta) - \sin^2(\beta+\theta) \\ &= \sin\{(\alpha+\theta)+(\beta+\theta)\} \sin\{(\alpha+\theta)-(\beta+\theta)\} \\ &= \sin(\alpha+\beta+2\theta) \sin(\alpha-\beta). \end{aligned}$$ F. M. GOLDNER.

1870. *Geometrical proof of* $v^2 = u^2 + 2fs$.

Boys sometimes ask whether there is a geometrical proof of this formula, having been shown the v, t diagram proofs of the other formulae for constant acceleration.

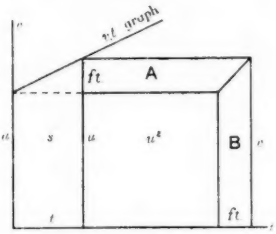


FIG. 1.

From the diagram, $v^2 - u^2$ = trapezium A + trapezium B. Each of these trapezia has parallel sides u and v, but width ft instead of t as in the s diagram. Each therefore represents fs. H. V. S.

1871. *Geometrical proof of the addition theorem for the tangent.*
This was suggested by a boy's treatment of a numerical case.

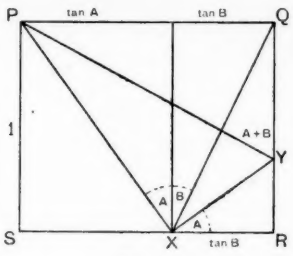


FIG. 2.

The figure shows the argument. We have a rectangle of unit height, and PQ is $\tan A + \tan B$. Then XY is perpendicular to PX, and so $PQYX$ is a cyclic quadrilateral. Hence

$$\angle PYQ = \angle PXQ = A + B.$$

Thence $$\tan (A+B) = PQ/QY\ ;$$

but $$QY = QR - YR = 1 - XR \tan A$$
$$= 1 - \tan A \tan B,$$

and the result is proved. H. V. S.

1872. *What is a trapezium?* (See Note 1798.)

I wonder if Mr. Siddons has seen the illuminating definitions in the *Concise Oxford Dictionary* :

Trapezium. Four-sided plane figure of which no two (or only two . . .) sides are parallel.

Trapezoid. Four-sided plane figure of which only two (or no two . . .) sides are parallel. G. A. C.

1873. *Real-life mathematics.*

Why the rate of exchange should be fixed to ensure our seeing *eye to eye* with our American Allies :

$$\begin{aligned} i^i &= (\cos \tfrac{1}{2}\pi + i \sin \tfrac{1}{2}\pi)^i \\ &= \cos \tfrac{1}{2}i\pi + i \sin \tfrac{1}{2}i\pi \\ &= e^{i(\frac{1}{2}i\pi)} \\ &= e^{-\frac{1}{2}\pi} \\ &= 0{\cdot}02788. \end{aligned}$$

Thus $£i^i = £0{\cdot}2079 = 4$s. 2d. to the nearest farthing
$= \$1$. G. A. C.

1874. *A modification of the Newton-Raphson method of approximation.*

1. If $x = a$ is an approximation to a root of an equation $f(x) = 0$, let $x = a + h$; then $f(a+h) = 0$, so that we have an equation for h,

$$f(a) + hf'(a) + h^2 f''(a)/2! + \ldots = 0.$$

It is convenient to divide throughout by $f'(a)$ so that

$$\alpha + h + \beta h^2 + \beta\gamma h^3 + \ldots = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

where $\alpha = f(a)/f'(a)$, $\beta = f''(a)/2f'(a)$, $\gamma = f'''(a)/3f''(a)$, etc.,

we may solve (i) for h by successive approximation or, alternatively, by using a formula for the reversion of power series,* from which it follows that $-h$ is the coefficient of $1/h$ in the expansion in powers of h of

$$\log(\alpha/h + 1 + \beta h + \beta\gamma h^2 + \ldots).$$

Hence

$$-h = \alpha + \alpha^2\beta + \alpha^3\beta(2\beta - \gamma) + \alpha^4\beta\{\gamma\delta + 5\beta(\beta - \gamma)\} + \ldots . \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

Successive approximations for h follow directly from (ii).

The first approximation, $h = -\alpha$, is Newton's formula. $\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$

The second approximation is

$$-h = \alpha(1 + \alpha\beta) \doteq \alpha/(1 - \alpha\beta);$$

hence

$$-1/h \doteq (1 - \alpha\beta)/\alpha = 1/\alpha - \beta.\dagger \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

Although Newton's formula, $h = -\alpha$, is well known and is in common use, the considerably closer approximation (iv) seems to merit consideration. It may often happen that the calculation of β is convenient, and once a good value of β has been obtained, it may be retained for use in the succeeding stages. In Newton's formula (iii) the error is of the order of h^2, while in (iv) the error is of the order of h^3.

2. We may derive from (ii) a further approximation which, while still comparatively simple, usually yields a much better approximation than (iv). If the terms involving α^4 and higher powers of α are omitted we obtain

$$-h = \alpha + \alpha^2\beta + \alpha^3\beta(2\beta - \gamma)$$

and putting

$$-h/\alpha = 1 + \alpha\beta + \alpha^2\beta(2\beta - \gamma) \doteq (1 + p\alpha)/(1 + q\alpha)$$

we obtain

$$p = -(\beta - \gamma) \text{ and } q = -(2\beta - \gamma),$$

whence

$$-\alpha/h \doteq 1 - \alpha\beta/\{1 + \gamma\alpha - \alpha\beta\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}$$

If in this equation we neglect $\alpha\gamma$ in comparison with 1 and for convenience write $r = 1/\beta = 2f'(a)/f''(a)$, we obtain

$$-1/h \doteq 1/\alpha - 1/(r - \alpha). \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(vi)}$$

While the error in (vi) is still of the order of h^3, it will usually give a closer approximation than either (ii) or (iv).

3. As an example to show the relative closeness of the approximations, we calculate $\sqrt[3]{20}$ as a root of the equation $f(x) \equiv x^3 - 20 = 0$, taking the first approximation a to be 3. It is found that $\alpha = 7/27$, $\beta = 1/3$ and $\gamma = 1/9$. By tables, $\sqrt[3]{20} = 2{\cdot}71441$.

Formula	Value	Error
(iii)	2·74075	·0263
(iv)	2·71622	·0018
(vi)	2·71366	·0007
(v)	2·71456	·0001

W. R. ANDRESS.

* Note 1657, *Gazette*, XXVII (May, 1943), p. 92.

† This formula is equivalent to one given recently by H. W. Richmond: *Journal L.M.S.*, No. 73, Jan. 1944, p. 32; *Edinburgh Math. Notes*, Sept. 1944, p. 7.

1875. *On Note* 1741.

In what follows, a modification and extension of the formula given by Mr. R. H. Birch in Note 1741 is indicated.

Mr. Birch gives the approximation :

$$(1+x)^{1/n} \simeq \frac{n\{(4+2x)^2 - x^2\} + 2x(4+2x)}{n\{(4+2x)^2 - x^2\} - 2x(4+2x)}, \quad \text{.....(i)}$$

or, if we write $z = x/(2+x)$, $x = 2z/(1-z)$,

$$\left(\frac{1+z}{1-z}\right)^{1/n} \simeq \frac{n(4-z^2)+4z}{n(4-z^2)-4z}. \quad \text{.....(ii)}$$

I should like to point out that this approximation could be much improved by changing the 4's in (ii) into 3's. First, however, I give a rather better approximation of the type (ii).

By expanding the left-hand side, namely, $(1+z)^{1/n}/(1-z)^{1/n}$, using the binomial theorem, and dividing, we obtain

$$\left(\frac{1+z}{1-z}\right)^{1/n} = 1 + \frac{2z}{n} + \frac{2z^2}{n^2} + \frac{2(n^2+2)z^3}{3n^3} + \dots \quad \text{.....(iii)}$$

Comparing this with the series obtained from

$$\frac{1+\alpha z+\beta z^2}{1-\alpha z+\beta z^2} = 1 + 2\alpha z + 2\alpha^2 z^2 + 2\alpha(\alpha^2-\beta)z^3 + \dots, \quad \text{.....(iv)}$$

we obtain in succession

$$\alpha = 1/n, \quad \alpha^2 = 1/n^2, \quad 2\alpha(\alpha^2-\beta) = 2(n^2+2)/3n^3,$$

whence $\beta = (1-n^2)/3n^2$. Thus, comparing the left-hand sides of (iii) and (iv), we find

$$\left(\frac{1+z}{1-z}\right)^{1/n} \simeq \frac{3n^2+3nz+(1-n^2)z^2}{3n^2-3nz+(1-n^2)z^2}. \quad \text{.....(v)}$$

This would give an error term of the order of z^4.

If, however, n is large we may, in the coefficients of z^2, neglect 1 in comparison with n^2, and so obtain

$$\left(\frac{1+z}{1-z}\right)^{1/n} \simeq \frac{n(3-z^2)+3z}{n(3-z^2)-3z}. \quad \text{.....(vi)}$$

This is of the same form as (ii) but should yield a smaller error, for in (ii) the error is of the order of $z^3(4-n^2)/6n^3$, and in (vi) it is of the order of $z^3/12n^3$. So, for small z and large n, (vi) should be a particularly good approximation, and (v) should be even better. W. R. ANDRESS.

1876. *On Note* 1775.

The calculation of logarithms by an elementary method giving three-place accuracy easily, but capable of giving any accuracy desired, is discussed on pp. 50–52 of the Association's *Report on the Teaching of Algebra in Schools*, 1934. C. O. T.

1877. *Sir Percy Nunn's "key theorem" in geometry.*

In the revised version of Sir Percy Nunn's sequence of theorems, the "key theorem" alluded to in the obituary notice (*Gazette*, XXIX, February, 1945, p. 2) became No. 13. This is printed below, prefaced by No. 12 to make the sequence of the argument clear.

Th. 12. *If two angles of a triangle are equal to corresponding angles of another triangle, the third angles of the triangles are also equal.*

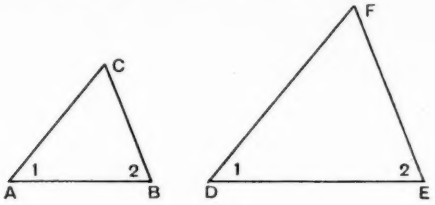


FIG. 1.

Given : Two triangles ABC, DEF in which $\angle A = \angle D$, $\angle B = \angle E$.

To prove : $\angle C = \angle F$.

Proof. By the Principle of Similarity, a triangle equiangular with ABC can be drawn upon DE in such a way that DE corresponds to AB. If the triangle is drawn, sides lying along DF and EF must be parts of it; for DF and EF are the only lines that make the angles at D and E equal to those at A and B. But lines drawn along DF and EF must cross at F and can meet nowhere else.

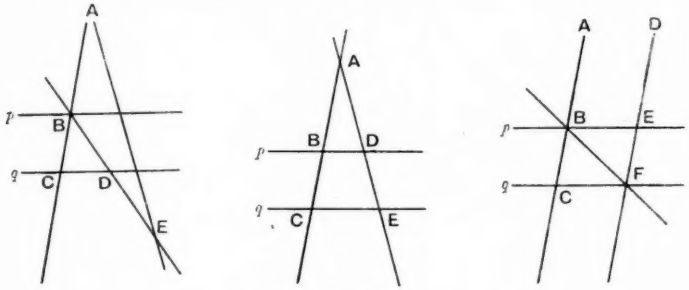
Hence the triangle on DE equiangular with the triangle ABC is identical with DEF; that is, the triangles ABC, DEF are equiangular and $\angle C = \angle F$.

Q.E.D.

Th. 13. *If any transversal cuts two lines at the same angle,* any other transversal will also cut them at the same angle.*

Given that a transversal ABC cuts the lines p and q at the same angle at the points B and C.

To prove that any transversal will also cut them at the same angle. There are three cases to be considered separately.



FIGS. 2, 3, 4.

Case I. When the second transversal ADE which cuts p and q at D and E meets ABC in a point A outside the region between p and q (Fig. 2).

Proof. In the triangles ABD, ACE, $\angle B = \angle C$ and $\angle A$ belongs to both. Thus the third angle at D is equal to the third angle at E (Th. 12); that is, ADE cuts p and q at the same angle.

* This phrase implies the equality of corresponding angles.

Case II. When the second transversal, BDE, cuts ABC on one of the lines p, q (Fig. 3).

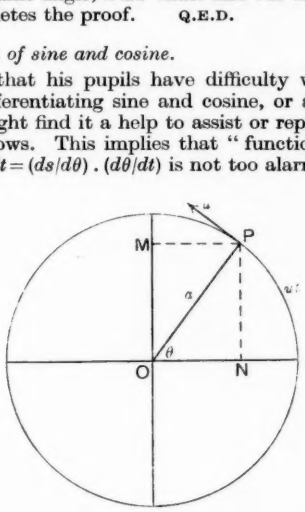
Proof. On ABC take a point A and on BDE a point E, both outside the region between p and q. Then the line AE is a transversal to p and q which cuts both the other transversals in points outside the region between p and q. Now since ABC cuts p, q at the same angle, AE must also cut them at the same angle (Case I); and since AE cuts them at the same angle, BE must also cut them at the same angle (Case I).

Case III. When the second transversal DEF which cuts the lines at E and F is parallel to ABC (Fig. 4).

Proof. Join BF. Then since ABC cuts the lines at the same angle, the transversal BF must also cut them at the same angle (Case II). And since BF cuts them at the same angle, DEF must also cut them at the same angle (Case II). This completes the proof. Q.E.D. C. O. T.

1878. *Differentiation of sine and cosine.*

Anyone who finds that his pupils have difficulty with the wherefore of radian measure in differentiating sine and cosine, or are bothered with the limit of $(\sin \frac{1}{2}\theta)/\frac{1}{2}\theta$, might find it a help to assist or replace the usual method by the one which follows. This implies that "function of a function" has been done so that $ds/dt = (ds/d\theta) \,.\, (d\theta/dt)$ is not too alarming.



In the usual figure for simple harmonic motion we have

for M: distance $= a \sin \theta$; velocity $= u \cos \theta$,

for N: distance $= a \cos \theta$; velocity $= -u \sin \theta$.

Thus using the formula

$$\frac{ds}{dt} = \frac{ds}{d\theta} \,.\, \frac{d\theta}{dt},$$

and noticing that if θ is in *radians*, $\theta = ut/a$ and so $d\theta/dt = u/a$, we get

$$u \cos \theta = \frac{d}{d\theta}(a \sin \theta) \,.\, \frac{u}{a},$$

and

$$-u \sin \theta = \frac{d}{d\theta}(a \cos \theta) \,.\, \frac{u}{a},$$

which give the desired results.

This method seems to give a different point of view on the need for using radian measure.

If the angle were x°, the arc would be $2\pi a \,.\, x/360 = ut$, so that

$$dx/dt = u \,.\, 180/\pi a,$$

and the first equation becomes

$$u \cos x^\circ = \frac{d}{dx}(a \sin x^\circ) \,.\, \frac{180u}{\pi a},$$

or

$$\frac{d}{dx}(\sin x^\circ) = \frac{\pi}{180} \,.\, \cos x^\circ.$$

Has anyone been using this method, or is it a novelty? C. O. T.

1879. $\log_{10} 1{\cdot}024$ *again.*

1. Dr. Carmody's approximation to $\log_{10}(x+\delta x)$ in Note 1774 (*Gazette*, Dec. 1944, pp. 199-200) appears to be based on the reciprocal of the arithmetic mean (A) of x and $x+\delta x$.

If $\delta x/x \equiv \epsilon$, we have

$$\log_e (x+\delta x) - \log_e x = \log_e (1+\epsilon)$$
$$= \epsilon - \tfrac{1}{2}\epsilon^2 + \tfrac{1}{3}\epsilon^3 - \tfrac{1}{4}\epsilon^4 + \tfrac{1}{5}\epsilon^5 - \tfrac{1}{6}\epsilon^6 + \tfrac{1}{7}\epsilon^7 - \tfrac{1}{8}\epsilon^8 + \dots \quad \dots(1)$$

and $\delta x/(x + \tfrac{1}{2}\delta x) = \epsilon(1+\tfrac{1}{2}\epsilon)^{-1}$

$$= \epsilon - \tfrac{1}{2}\epsilon^2 + \tfrac{1}{4}\epsilon^3 - \tfrac{1}{8}\epsilon^4 + \tfrac{1}{16}\epsilon^5 - \tfrac{1}{32}\epsilon^6 + \tfrac{1}{64}\epsilon^7 - \tfrac{1}{128}\epsilon^8 + \dots \quad \dots(2)$$

Hence $\log_e (1+\epsilon) = \epsilon(1+\tfrac{1}{2}\epsilon)^{-1} + \tfrac{1}{12}\epsilon^3 - \tfrac{1}{8}\epsilon^4 + \tfrac{11}{80}\epsilon^5 - \tfrac{13}{96}\epsilon^6 \dots \quad \dots(3)$

When $x = 1000$ and $\delta x = 24$,

$$\mu\epsilon^3/12 = {\cdot}4343 \times 1{\cdot}152 \times 10^{-6} = 5 \times 10^{-7},$$

and we obtain $\log_{10} 2 = {\cdot}3010300$.

2. We may construct a similar rule by using the reciprocal of the harmonic mean (H). We have

$$\delta x(x+\tfrac{1}{2}\delta x)/x(x+\delta x) = \epsilon(1+\tfrac{1}{2}\epsilon)(1+\epsilon)^{-1}$$
$$= \epsilon - \tfrac{1}{2}\epsilon^2 + \tfrac{1}{2}\epsilon^3 - \tfrac{1}{2}\epsilon^4 + \tfrac{1}{2}\epsilon^5 - \tfrac{1}{2}\epsilon^6 + \tfrac{1}{2}\epsilon^7 - \tfrac{1}{2}\epsilon^8 + \dots, \quad \dots(4)$$

so that

$$\log_e (1+\epsilon) = \epsilon(1+\tfrac{1}{2}\epsilon)(1+\epsilon)^{-1} - \tfrac{1}{8}\epsilon^3 + \tfrac{1}{4}\epsilon^4 - \tfrac{3}{10}\epsilon^5 + \tfrac{1}{3}\epsilon^6 + \dots \quad \dots(5)$$

This is not so good as the former rule, but may be combined with it to construct a much better one. Doubling equation (3) and adding it to (5):

$$3\log_e (1+\epsilon) = 2\epsilon(1+\tfrac{1}{2}\epsilon) + \epsilon(1+\tfrac{1}{2}\epsilon)(1+\epsilon)^{-1} - \tfrac{1}{40}\epsilon^5 + \tfrac{1}{16}\epsilon^6 + \dots,$$

or

$$\log_{10}(x+\delta x) = \log_{10} x + \tfrac{1}{3}\mu\delta x\left(\frac{2}{A} + \frac{1}{H}\right) - \frac{\mu}{120}\epsilon^5 + \frac{\mu}{48}\epsilon^6 \dots$$

For example, if $\epsilon = {\cdot}024$,

$$\log_{10}(1{\cdot}024) = \mu\left(\frac{4}{253} + \frac{{\cdot}253}{32}\right),$$

with error $-\mu({\cdot}024)^5/120$ or $-2{\cdot}88 \times 10^{-11}$.

Taking $\log_e 10 = 2{\cdot}3025850930$, this gives

$$\log_{10} 2 = {\cdot}30102999566.$$

3. We may also combine with these rules a similar one based on the geometric mean (G) in order to construct a formula of quite remarkable accuracy.

Here $\delta x/\sqrt{\{x(x+\delta x)\}} = \epsilon(1+\epsilon)^{-\frac{1}{2}}$

$= \epsilon - \frac{1}{2}\epsilon^2 + \frac{3}{8}\epsilon^3 - \frac{5}{16}\epsilon^4 + \frac{35}{128}\epsilon^5 - \frac{63}{256}\epsilon^6 + \frac{231}{1024}\epsilon^7 - \frac{429}{2048}\epsilon^8 + \ldots$ .....(6)

Multiply (1) by 15, (2) by -2, (4) by 3, (6) by -16, and add all the results. The right-hand side reduces to

$$\epsilon^7/448 - \epsilon^8/128.$$

Thus $$\log_{10}(x+\delta x) = \log_{10} x + \frac{1}{15}\mu\,\delta x\left(\frac{2}{A} - \frac{3}{H} + \frac{16}{G}\right) + \mu\left(\frac{\epsilon^7}{6720} - \frac{\epsilon^8}{1920}\right).$$

If $\epsilon = \cdot 024$,

$$\log_{10} 1\cdot 024 = \frac{\mu}{5}\left(\frac{4}{253} - \frac{\cdot 759}{32} + \frac{\sqrt{160}}{100}\right)$$

$$= \mu \times \cdot 0237165266173154,$$

with error less than $\mu(\cdot 024)^7/6720$ or 3×10^{-16}.

Taking $\log_e 10 = 2\cdot 302585092994046$ (H. S. Uhler, *Original Tables to 137 Decimal Places . . .*), we obtain

$$\log_{10} 1\cdot 024 = \cdot 010299956639811.$$

According to A. J. Thompson, *Logarithmetica Britannica*, the last figure should be 2. Thus

$$\log_{10} 2 = \cdot 301029995663981.$$

4. This does not agree with Mr. Birch's figures in Note 1776 of the same *Gazette*. It turns out that the value of μ he quotes (p. 202) does not go far enough for his purpose, while his "error" should be $1\cdot 8 \times 10^{-11}$ and not $1\cdot 8 \times 10^{-10}$.

Taking $\mu = 1/2\cdot 3025850930$, we now get from his formula

$$\log_{10} 1\cdot 024 = \cdot 010299956622 + 1\cdot 8 \times 10^{-11},$$

or $$\log_{10} 2 = \cdot 3010299956640.$$

5. We may compare § 3 with some observations in Sir C. V. Boys' book, *The Natural Logarithm*, noted by W. C. Fletcher in his review of this book in the *Gazette*, XIX, July 1935, p. 243. The results may be harmonised by the identity

$$\frac{1}{A} + \frac{1}{H} - \frac{2}{G} - \left(\frac{1}{H_2} - \frac{1}{A_2}\right) = \frac{(b^{\frac{1}{4}} - a^{\frac{1}{4}})^8}{4ab(a+b)(a^2+6ab+b^2)},$$

where A, H, G are means of a and b, and A_2, H_2 those of A and H.

For example, when $a = 1000$ and $b = 1024$, the above expression is less than 10^{-20}.

N. M. GIBBINS.

1880. *On Note* 1762.

Duncan C. Fraser, in his paper "Elementary Notes on Simple Summations and Summations of Products" shows that Pascal's Triangle gives both the coefficients of the expansion of $(1+1)^n$ and, by reading along the diagonals in the usual layout,* the coefficients of the expansion $(1-1)^{-n}$.

Further, as would be expected, if the triangle be extended backwards by the same rule (so that a wedge of zeros appears on the left) the horizontal lines give the coefficients of $(1+1)^{-n}$, and the diagonals the coefficients of $(1-1)^n$.

* Fraser is considering the triangle as an extension, to the left of the tabular values, of the ordinary difference table. The leading diagonal of units in the ordinary arrangement is therefore vertical in his. The remarks above apply to the former.

This paper appeared in the *Transactions of the Faculty of Actuaries*, probably around 1930. The reprint I have is undated, and is without any volume or page numbers that might give a clue, which is rather disturbing in data issued from actuarial sources.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & -4 & & 1 & & & & & \\
 & & & & 6 & & -3 & & 1 & & & & \\
 & & & -4 & & 3 & & -2 & & 1 & & & \\
 & & 1 & & -1 & & 1 & & -1 & & 1 & & \\
 & . & & . & & . & & . & & . & & 1 & \\
 & & . & & . & & . & & . & & 1 & & 1 \\
 & . & & . & & . & & . & & 1 & & 2 & & 1 \\
 & & . & & . & & . & & 1 & & 3 & & 3 & & 1 \\
 & . & & . & & . & & 1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

R. A. FAIRTHORNE

1881. *Combined proof of the angle-bisector theorem and the tangent formula.*

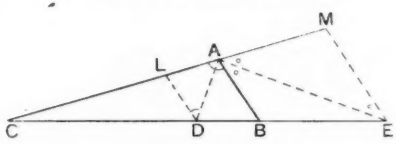
1. Let AD, AE be the internal and external bisectors of the angle BAC, and let DL, EM, parallel to BA, meet CA in L, M.

Then $$BD : DC = AL : LC.$$

But $$\angle LDA = \angle DAB = \angle LAD,$$

whence $$LD = LA,$$

and so $$BD : DC = LD : LC.$$



The triangles CLD, CAB are similar, and so

$$LD : DC = BA : AC.$$

Hence $$BD : DC = BA : AC.$$

The proof for the external bisector is similar.

2. With the usual notation, from above, we have

$$CL : LA = b : c,$$

and so $$LA = bc/(b+c);$$

similarly, $$AM = bc/(b-c).$$

But $$AD = 2AL \cos \tfrac{1}{2}A = \frac{2bc}{b+c} \sin \tfrac{1}{2}(B+C),$$

and $$AE = 2AM \cos \tfrac{1}{2}MAB = \frac{2bc}{b-c} \cos \tfrac{1}{2}(B+C).$$

Further, $\angle DAE = 90°$, and $\angle AED = \angle CBA - \angle BAE$

$$= B - \tfrac{1}{2}(B+C) = \tfrac{1}{2}(B-C).$$

Thus $$\tan \tfrac{1}{2}(B-C) = AD/AE$$
$$= \frac{b-c}{b+c} \tan \tfrac{1}{2}(B+C).$$

3. *Corollary.* Assuming the formulae for $\sin \frac{1}{2}A$, $\cos \frac{1}{2}A$,

$$AD = 2\sqrt{\{bcs(s-a)\}}/(b+c),$$
$$AE = 2\sqrt{\{bc(s-b)(s-c)\}}/(b \sim c).$$ D. F. FERGUSON.

1882. *Residual types of partitions of* " 0 " *into four cubes.*

It is fairly well known that when a, b, c, d are four integers positive or negative such that

$$a^3 + b^3 + c^3 + d^3 = 0,$$

then $a+b+c+d$ is always a multiple of 6. This is obvious if we recollect that

$$x^3 \equiv x \pmod 6, \text{ when } x = 0, 1, 2, 3, 4, 5,$$

and hence

$$a^3 + b^3 + c^3 + d^3 \equiv a + b + c + d \pmod 6.$$

We now examine the types of residues of a, b, c, d whose sum is zero. We exclude naturally those cases where a, b, c, d have a common factor. We regard the residual types of a, b, c, d and of $-a$, $-b$, $-c$, $-d$ as equivalent. Under the above convention, the following different cases have to be considered :

1. All residues different : The only possible sets are (0, 1, 2, 3) and (1, 2, 4, 5), since (0, 3, 4, 5) is by our convention equivalent to (0, 3, 2, 1).
2. All residues alike : This case impossible.
3. Three alike, one different : There is only one case of this kind, (1, 1, 1, 3).
4. Two pairs of like residues : There are only two cases, (1, 2, 1, 2) and (1, 5, 1, 5).
5. Two different and two alike : There are five cases of this kind, (0, 0, 1, 5) ; (0, 1, 1, 4) ; (1, 3, 3, 5) ; (2, 2, 3, 5) and (2, 3, 3, 4).

Using index symbolism for repeated parts, we write below the only possible types of partitions, just *ten* in number, enumerated in dictionary order :

TYPES.	ILLUSTRATIONS.
I. $(0^2\ 1\ 5)$.	$12^3 + 19^3 + 53^3 + (-54)^3 = 0$.
II. $(0\ 1\ 2\ 3)$.	$46^3 - 37^3 - 3^3 = 6^6$.*
This type happens to be of most frequent occurrence.	
III. $(0\ 1^2\ 4)$.	No example available.
IV. $(1^3\ 3)$.	No example available.
V. $(1^2\ 2^2)$.	$4^3 + 17^3 + 22^3 + (-25)^3 = 0$.
VI. $(1^2\ 5^2)$.	$13^3 + 65^3 + 121^3 + (-127)^3 = 0$.
VII. $(1\ 2\ 4\ 5)$.	$791^3 + 812^3 = 1010^3 - 1$.*
VIII. $(1\ 3^2\ 5)$.	$11^3 + 15^3 + 27^3 + (-29)^3 = 0$.
IX. $(2^2\ 3\ 5)$.	No example available.
X. $(2\ 3^2\ 4)$.	$33^3 + 70^3 + 92^3 + (-105)^3 = 0$.

From the above table we may infer the following result :

* These examples are due to Ramanujan (vide *Collected Papers*, p. 331).

Theorem. If the sum of four cubes vanishes, then at least one of them is of the form $(6m \pm 1)$ or $(6m + 3)$, where m is an integer.

N.B.—The absence of illustrations for the Types III, IV and IX, even after a careful search * among the existing examples, makes one feel that they are probably non-existent. Will any interested reader probe more deeply into these cases?

In conclusion I thank Prof. A. A. Krishnaswami Ayyangar for his guidance in this investigation.

A. K. SRINIVASAN.

1883. *A problem in integration.*

Let u, v, w be three functions of the independent variables x, y, and let a homogeneous function $\phi(u, v, w)$ of degree n become $S(x, y)$ when the variables are changed. Then on the curve $S = 0$ we have

$$\frac{dx}{S_y} = -\frac{dy}{S_x} = \frac{du}{\begin{vmatrix} u_x & S_x \\ u_y & S_y \end{vmatrix}} = \frac{dv}{\begin{vmatrix} v_x & S_x \\ v_y & S_y \end{vmatrix}}$$

$$= \frac{u\,dv - v\,du}{\begin{vmatrix} u_x & v_x & S_x \\ u_y & v_y & S_y \\ u & v & 0 \end{vmatrix}} = \frac{u\,dv - v\,du}{\begin{vmatrix} u_x & v_x & u_x\phi_u + v_x\phi_v + w_x\phi_w \\ u_y & v_y & u_y\phi_u + v_y\phi_v + w_y\phi_w \\ u & v & u\phi_u + v\phi_v + w\phi_w \end{vmatrix}}$$

since $u\phi_u + v\phi_v + w\phi_w$ is $n\phi$ identically and is therefore zero on the curve. Thus

$$\frac{dx}{S_y} = \frac{u\,dv - v\,du}{\mu\phi_w},$$

where

$$\mu = \begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u & v & w \end{vmatrix}.$$

This result is general. If u, v, w are linear functions of x and y, the elements of the first two rows in μ are the coefficients of x and y in the three functions and μ reduces immediately to a constant. On the other hand, if

$$\phi(u, v, w) = uv - w^2,$$

then $\phi_w = -2w$, and on the curve we have

$$\frac{dx}{\frac{1}{2}wS_y} = \frac{1}{\mu}\left(\frac{du}{u} - \frac{dv}{v}\right),$$

whatever the nature of u, v, w. With the double specialisation we have a result from the Tripos of 1902, which will be found as an exercise in *Pure Mathematics* (p. 284, ex. 61); no solution is in the current edition, and the solution published in the first edition leaves us completely in the dark as to how the result was in fact discovered.

E. H. N.

1884. *Technical mathematics.*

With regard to Professor Neville's remarks on the discussion of the above subject (*Math. Gazette*, October, 1945, p. 160), I have often said that, when

* I have tested the "Tables of Partition" of Russel and Gwyther in vain for the missing types (*Math. Gaz.*, XXI, pp. 34, 35).

an engineer wants help from a mathematician, it is essential that (i) the engineer should know enough mathematics to state his problem, and (ii) the mathematician should know enough about engineering to understand the engineer's statement of his problem.

I had a simple illustration of this towards the end of the last war.

In 1918 I received a 16-page letter from an artillery officer asking for my help in the solution of a gunnery problem. I could make neither head nor tail of what the man meant, and I said that I was incapable of dealing with his problem and suggested that he should write to Woolwich. In reply I had another 16-page letter which again was quite unintelligible, and in which my correspondent said that he was sure that I could deal with the question but that he could not explain what he wanted. Luckily he enclosed a sheet on which he had made some calculations; from that I spotted what his problem was. He had an angle θ expressed approximately in powers of another small angle α, say $\theta = \alpha + a\alpha^2 + b\alpha^3$; he wanted to express α in terms of θ. This job was done in a minute or two. One not uninteresting fact was that he had found an expression for α in terms of θ, but it did not work—his only mistake was that he had left out a factor $\pi/180$ in the coefficient of one term.

In contrast to the above, two engineers have submitted problems to me in the last year. The first wrote a very clear description of what he wanted and sent me the Government specification; his difficulty proved to be that one essential piece of data was missing from the specification. Incidentally this led to an order for $1\frac{1}{2}$ million pieces of metal of a complicated shape. The second engineer brought the firm's mathematician with him. After the engineer had given me a clear explanation of the mechanism concerned, he turned on his mathematician to show me his attempt at the solution.

A. W. S.

1885. *Hadamard's Theorem.*

Rev. A. Lai-Fook has pointed out that the argument of my note on Symmetric Matrices and Quadratic Forms (*M.G.*, July, 1945) furnishes a simple proof of Hadamard's Theorem, that the square of a real determinant (or the square of the absolute value of a complex determinant) does not exceed the product of the "scalar squares" of the column-vectors, and is equal to this product when the column-vectors are mutually perpendicular.

If M is a real square matrix of order and rank n, the matrix $A = M'M$ is positive-definite, and its element a_{pq} is the scalar product of the pth and qth column-vectors of M. Accordingly the proposition reduces to the statement that the determinant of a positive-definite form does not exceed the product of its diagonal elements, and is equal to this product only when there is no non-zero element outside the diagonal.

If there is such a non-zero element, we can take it to be a_{12}. We can write $x'Ax = \Sigma b_p y_p^2$ by a unimodular transformation, where

$$b_1 = a_{11}, \; b_2 = a_{22} - a_{12}^2/a_{11},$$

and it is easily seen that $b_p \leqslant a_{pp}$ for $p > 2$. Accordingly we have

$$|A| = \prod_1^n b_p < \Pi a_{pp}.$$

When the non-diagonal elements are all zero, we clearly have equality. If M is complex, we consider similarly the positive-definite Hermitian matrix $\overline{M}'M$.

Hadamard's Theorem is the extension to n dimensions of the theorem that the volume of a parallelepiped does not exceed the product of the lengths of the edges meeting in a corner, and is equal to it when these edges are perpendicular.

M. F. EGAN.

1886. *On the three-cusped hypocycloid—Addendum* *.

Of course, as soon as you had mentioned the identity of the Simson lines with the asymptotes of the equilateral hyperbolas—a fact which I had completely overlooked—I thought of a geometrical way of realising it.

This can be done rather simply by the following considerations. Considering the inscribed triangle ABC, with the orthocentre D, let us take the Simson lines L, L' of two diametrically opposite points M, M' of the circumcircle. These are orthogonal to each other. They go respectively through the midpoints m, m' of DM, DM' : that is, through two diametrically opposite points of the nine-point circle **C** ; and as they are orthogonal, their intersection w again lies on the same circle. Then :

(1) The midpoint between the projections† of M, M' on to any side of the triangle, say AB, is the midpoint of AB : therefore the equilateral hyperbola having L, L' for its asymptotes which goes through A also goes through B and similarly through C (therefore through D).

(2) As to the direct proof that we thus get a hypocycloid, we get it by considering the angle ϕ by which m rotates around the centre of **C**, that is the angle by which M rotates on the circumcircle.‡ Now, if D_1 is the symmetrical point of D with respect to BC, the line D_1M rotates by $\frac{1}{2}\phi$ and the Simson line, whose direction is symmetrical of D_1M with respect to BC, rotates by $\lambda\phi$ with

$$\lambda = -\tfrac{1}{2}.$$

Arguing as I do in my paper, we get the same conclusions as I did.

(3) In the aforementioned paper, λ has the value $+\frac{1}{4}$. Now, if through a point m variable on a circle **C** and defined by its argument ϕ, we draw a line L of argument $\lambda\phi$, its envelope will be an epi- or hypo-cycloid. But the same line L intersects **C** at another point m_1 (argument ϕ_1) and $\lambda\phi$ will be equal to $\lambda_1\phi_1$, so that we obtain the same hypocycloid if we start from the same **C** and change λ into λ_1. It is easy to see that the relation between λ and λ_1 is

$$\frac{1}{\lambda} + \frac{1}{\lambda_1} = 2,$$

a relation which is satisfied by the two numbers $-\frac{1}{2}$ and $+\frac{1}{4}$.

This double generation of the hypo- (or epi-) cycloid as an envelope of tangents precisely corresponds to its double generation as a locus of points obtained by keeping R and changing r into $R - r$ (R, radius of the fixed circle ; r, radius of the rolling circle).

J. HADAMARD.

* I ventured recently to recall to Professor Hadamard the fact that the asymptotes of any pencil of equilateral hyperbolas are also Simson lines of each triangle whose vertices are three base-points of the pencil. This provides an alternative approach to the generation of the hypocycloid treated in his paper in *Math. Gazette*, **29**, 66–7, 1945.

Professor Hadamard thereupon sent me the following very concise proof of the identity of the asymptotes and Simpson lines and of the fact that they envelop a hypocycloid. He has now very kindly given me permission to communicate his letter to the *Gazette*.

The two small footnotes have been inserted by me.—W. H. MCCREA.

† These projections are on L, L', respectively, by definition of the Simpson lines.

‡ MM' and mm' being parallel diameters of the two circles.

1480. P's for Protection Potatoes afford ;
O's for the Ounces of Energy stored.
—Ministry of Food advertisement. [Per Mr. A. J. L. Avery.]

REVIEWS.

The New York Mathematical Tables Project. III.

1. **Table of Arc sin x.** Pp. xx, 124. $7\frac{1}{4}'' \times 10\frac{1}{2}''$. 1945. 22s. 6d. (Columbia University Press, New York)

2. **Table of Arc tan x.** Pp. xxvi, 174. $8'' \times 10\frac{1}{2}''$. 1942. 20s.

3. **Tables of Sine, Cosine and Exponential Integrals.** Vol. I. Pp. xxviii, 450. $8'' \times 10\frac{1}{2}''$. 1940. 20s.

4. **Tables of Sine, Cosine and Exponential Integrals.** Vol. II. Pp. xxxviii, 230. $8'' \times 10\frac{1}{2}''$. 1940. 20s.

5. **Table of Sine and Cosine Integrals for Arguments from 10 to 100.** Pp. xxxii, 188. 1942. 20s.

(All the tables were prepared by the Mathematical Tables Project of the Work Projects Administration for the City of New York; technical director, Arnold N. Lowan. Items 2 to 5 were published by the Federal Works Agency, W.P.A. for the City of New York. British Agents for all volumes, Scientific Computing Service, 23 Bedford Square, London, W.C. 1.)

In two earlier articles (*Mathematical Gazette*, **29**, 1945, pp. 29-33, 86-87) six volumes of tables produced by New York M.T.P. have been described. The publication of a new table, the first listed above, provides an opportunity to describe also the closely related *Table of Arc tan x*, (2) above. The three other tables are the only other ones involving circular functions that have not been already described.

The important tables contained in individual volumes are as follows:

(1) The main table gives 12-decimal values of $\sin^{-1} x$ for

$$x = 0(0{\cdot}0001)0{\cdot}989(0{\cdot}00001)1.$$

Second differences, modified where necessary, are given for $x \leqslant 0{\cdot}999$ and suffice for interpolation, by means either of the Newton-Gregory formula, or of the Everett formula, throughout this range. For $0{\cdot}999 \leqslant x \leqslant 0{\cdot}9995$, δ^2 and δ^4 are given. There are no differences for $x > 0{\cdot}9995$, but instead an auxiliary table of $f(v)$ is given, where

$$\sin^{-1}(1-v) = \tfrac{1}{2}\pi - \sqrt{2v}f(v),$$

to 13 decimals for $v = 0(0{\cdot}00001)0{\cdot}0005$, with Δ and Δ^2.

(2) The main table gives 12-decimal values of $\tan^{-1} x$ for

$$x = 0(0{\cdot}001)7(0{\cdot}01)50(0{\cdot}1)300(1)2000(10)10000.$$

Second differences are given and suffice for full interpolation throughout the table.

(3) This gives 9-decimal values of

$$\mathrm{Si}\, x = \int_0^x \frac{\sin t}{t}\, dt, \qquad \mathrm{Ci}\, x = \int_\infty^x \frac{\cos t}{t}\, dt,$$

$$\mathrm{Ei}\, x = \int_{-\infty}^x \frac{e^t}{t}\, dt, \qquad -\mathrm{Ei}(-x) = \int_x^\infty \frac{e^{-t}}{t}\, dt,$$

for $x = 0(0{\cdot}0001)2$, and for $x = 0(0{\cdot}1)10$. The auxiliary functions

$$\mathrm{Ci}\, x - \ln x, \quad \mathrm{Ei}\, x - \ln x, \quad -\mathrm{Ei}(-x) + \ln x$$

are tabulated for $x = 0(0{\cdot}0001)0{\cdot}0099$, also to 9 decimals. Second differences are given everywhere, except for the main function when an auxiliary function is given, for all arguments when variation is fairly rapid or for every 25th or 50th argument when variation is slow.

(4) Table I in this volume gives Si x and Ci x to 10 decimals, Ei x to 10 figures and $-\mathrm{Ei}(-x)$ to 9 figures, for $x = 0(0{\cdot}001)10$. Second differences, for every 25th argument, are given throughout.

Table II gives Si $(p\pi \pm h)$, $p = 1, 2, 3$, and Ci $(\frac{1}{2}q\pi \pm h)$, $q = 1, 3, 5$, to 15 decimals for $h = 0(0{\cdot}0001)0{\cdot}01(0{\cdot}001)0{\cdot}05$; second differences and an indication of fourth differences are given.

Table III, reproduced by permission from the B.A. *Mathematical Tables* Vol. I, gives 10-decimal values of Si x and Ci x (with even differences up to the sixth, sufficient for accurate interpolation) for $x = 10(0{\cdot}1)20(0{\cdot}2)40$. It also gives 10-figure values of Ei x, and 14-decimal values of $-\mathrm{Ei}(-x)$, with second and fourth differences, for $x = 10(0{\cdot}1)15$.

(5) The main table in this volume gives 10-decimal values of Si x and Ci x with full second differences throughout, for $x = 10(0{\cdot}01)100$.

The various introductions are of considerable interest; all are on much the same plan, giving first a description of the function, or functions, tabulated and of their properties. This is followed by sections on interpolation and on the computation and checking of the tables. Finally, in each case, there is an excellent bibliography of earlier tables; and, in (3), (4) and (5), a list of references to applications, and graphs of various functions.

Each volume concludes with a number of minor tables, giving interpolation coefficients and multiples of $\frac{1}{2}\pi$ and, in some cases, other conversion tables.

The tables have been reproduced from typescript in every case, with steadily improving results. The latest volume (1), makes use of a more satisfactory style of figures, with very slight "heads" and "tails". The improvement is such that it seems worth while to suggest that further improvement might result from the use of slightly smaller figures, in the same style, to allow a little more space between lines. The use of printed forms on which the tables are typed has also led to noticeable improvement in the appearance of the tables in (1), (2) and (5). The latest volume, (1), is bound in red—an unimportant, but mildly disappointing, departure from what had appeared to be an established custom!

Tests of accuracy applied to the New York M.T.P. tables reviewed earlier suggest that it would be unprofitable to make any great effort to test the accuracy of the tables now reviewed. No such tests have been applied, but it is clear that there is every reason for confidence in the accuracy of the tables.

It seems worth while to make a few comments on certain points mentioned in the introductions to (1) and (2).

On p. xvi of (1) it is noted that the formula

$$\sin^{-1} x = \tfrac{1}{4}\pi + \tfrac{1}{2}\sin^{-1}(2x^2 - 1)$$

was used for $x = 0{\cdot}5(0{\cdot}01)0{\cdot}85$, when preparing the tables. This same formula may be used to reduce the number of cases where the auxiliary function $f(r)$ or the use of fourth difference interpolation, is necessary. Another such formula is

$$\sin^{-1} x = \tfrac{1}{3}\sin^{-1}(3x - 4x^3).$$

In table (2), although second differences are provided for interpolation, the

simplest method—especially if a calculating machine is available—seems to be that given at the foot of p. xv, that is by use of the formula

$$\tan^{-1}(x_0+\lambda)=\tan^{-1}x_0+u-\tfrac{1}{3}u^3,$$

in which the quantity $u=\lambda/\{1+x_0(x_0+\lambda)\}$ is very easy to evaluate, since x_0 may be chosen as the nearest tabular argument, so that λ may be written down at once. The last term nowhere exceeds 42 units in the twelfth decimal and may be obtained from a small auxiliary table (direct or critical). It seems a pity that such a table was not provided; it may be worth while to give one.

$10^3 . u$	$10^{12} . \frac{1}{3}u^3$	$10^3 . u$	$10^{12} . \frac{1}{3}u^3$	$10^3 . u$	$10^{12} . \frac{1}{3}u^3$
0·00000		0·35169		0·44563	
	0		15		30
·11447		·35959		·45061	
	1		16		31
·16509		·36717		·45548	
	2		17		32
·19574		·37444		·46025	
	3		18		33
·21897		·38144		·46493	
	4		19		34
0·23811		0·38819		0·46951	
	5		20		35
·25458		·39472		·47400	
	6		21		36
·26916		·40103		·47841	
	7		22		37
·28231		·40716		·48274	
	8		23		38
·29433		·41310		·48699	
	9		24		39
0·30545		0·41888		0·49117	
	10		25		40
·31581		·42450		·49528	
	11		26		41
·32554		·42998		·49933	
	12		27		42
·33471		·43532		·50331	
	13		28		43
·34341		·44054		·50722	
	14		29		44
0·35169		0·44563		0·51108	

In a fairly large class of cases—the evaluation of $\tan^{-1}(m/n)$, m and n being integers—interpolation may be avoided altogether. This is discussed in replies to a query in *Mathematical Tables and other Aids to Computation*, published by the National Research Council, Washington, D.C.; these will appear in forthcoming issues of the same journal. A single example may be given to illustrate the type of formula:

$$\tan^{-1}\frac{47}{70}=\tfrac{1}{2}\pi+\tan^{-1}0{\cdot}3-\tan^{-1}5{\cdot}205.$$

One further point may be noted. In (1), p. xii, modified differences are defined by the relation

$$\delta^{2*}u=\delta^2u-0{\cdot}183938^4u.$$

(Modified differences are described in B.A. *Mathematical Tables*, vol. 1 *, p. xi., or in *Interpolation and Allied Tables*, p. 928.) The factor usually adopted is 0·184 ; the factor 0·18393 is strictly more accurate, but the gain is entirely negligible in comparison with the residual fourth difference term needed to correct for the use of an approximate coefficient for the fourth difference. Although 0·184 is slightly less accurate, it is far more convenient when modified second differences are used, as they may be effectively used, in connection with fourth differences much exceeding the usual limit of 1,000 ; in this case the residual fourth difference correction is essential, and the factor 0·184 makes it possible to tabulate the residual fourth difference coefficient *exactly* with fewer decimals than does the factor 0·18393 (even if this also is assumed to be exact—although the best possible factor is, in fact, irrational, being $(3+\sqrt{2})/24$). It is to be hoped that the definite factor 0·184 can be adopted generally.

Such criticisms as have been made above are very minor ones, and should not be considered as detracting in any way from the merits of these volumes of tables, which are of outstanding importance and interest. Each is by far the most extensive table of the function or functions concerned, and the Table of Arc tan x, in particular, has filled a gap that had been noticeable for a considerable time ; no mathematical computing establishment can be considered fully equipped unless it has all of these volumes on its shelves.

J. C. P. MILLER.

Spherical trigonometry after the Cesàro method. By J. D. H. DONNAY. Pp. xi, 83. $1.75. 1945. (Interscience Publishers, N.Y.)

The method used to develop the subject-matter of this little book is based on stereographic projection. Taking advantage of the fundamental properties of the latter, any spherical triangle can be made to correspond to a plane triangle whose six elements are related to those of the spherical triangle together with the spherical excess. This process provides neat proofs of the sine, cosine and several other less important formulae. Unfortunately, the four parts formula is more stubborn.

The process of relating a spherical triangle to a plane triangle is attractive, and this book makes interesting reading for anyone unfamiliar with the method used. This approach does not, however, commend itself for the purpose of introducing students to spherical trigonometry. The opening chapter, on stereographic projection, requires careful reading, and when this is added to the second chapter, introducing the derived triangle, the total amount of preliminary work to be done more than offsets any subsequent advantage over the normal method. Spherical trigonometry is a subject whose purpose is largely utilitarian, and what a student requires above all else is a clear understanding of the meaning of sides and angles of a spherical triangle, and a knowledge of the sine, cosine, polar cosine and four parts formulae, together with Napier's rules. The fact that these are preceded in the present book by a chapter dealing with less important formulae tends to obscure their importance.

B. M. B.

Science in Progress. 4th Series. Edited by G. A. Baitsell. Pp. xvi, 331. 20s. 1945. (Yale University Press ; Humphrey Milford)

This volume, the fourth of a series, contains eleven articles written for the National Lecturerships of the Society of the Sigma Xi, which has as its object the encouragement of research in science. A rather large proportion of the subjects are biological, but physics and chemistry are represented.

* First edition only.

The only article which is essentially mathematical is one by the late Professor G. D. Birkhoff on the mathematical nature of physical theories. In this are discussed in a general manner some of the mathematical " models " of the universe, from Euclid to the dynamical world of Newton, the 4-vectors and tensors of Einstein and the quantum theory. The disadvantages of the various theories are explained and possible future developments considered.

The articles are written by specialists for non-specialists and their style is such that they can be read with interest by anyone whose knowledge of a particular subject is confined to little more than " knowing the language ". The paper and printing are of a quality unusual for the present time, although the latter is marred by the fact that the compositors appear to have no idea of handling mathematical formulae, particularly in one article. B. M. B.

The Development of Mathematical Logic and of Logical Positivism in Poland between the Two Wars. By Z. JORDAN. Pp. 47. 2s. 6d. 1945. (Polish Science and Learning, No. 6, Sir Humphrey Milford, Oxford University Press)

Many of those familiar with the work of Lukasiewicz, Lesniewski, Ajdukiewicz, Chwistek, Tarski and Kuratowski will be most interested to learn of the origins of the Polish School. Dr. Jordan, who wrote this booklet while on active service with the First Polish Armoured Division, is a philosopher from Poznan. In his opinion the real beginning was the appointment of Twardowski to Lwów in 1895 ; Twardowski exercised a wide and powerful influence far beyond the circle of professional philosophers, although his formal contributions to the subject have not been very extensive.

Among the notable facts revealed is that a course of two hours a week on the fundamentals of logic (and psychology) was prescribed for students in their last year at secondary schools in Poland. This was also the case in Austria. Here, we feel, is an example worth following ; the introduction of such a course or perhaps a similar one for first year students at the Universities would be of greatest advantage in the promotion of clearer thinking. There is at hand Tarski's *Introduction to Logic* (Oxford University Press, New York, 1941) as an excellent textbook.

The beginning of formal contribution to mathematical logic by the Polish School occurred about 1910. The earlier work was suggested by critical examinations of *Principia Mathematica*, and is represented by Chwistek's theory of types and Lukasiewicz's improvement in the *Principia* symbolism. After the end of 1914-1918 war we find contributions from Tarski and Kuratowski ; for instance, Kuratowski's work on " order " which may be recalled by his definition of an " ordered pair " (a, b) as the class consisting of two members : the class containing a and the class containing a and b.

Following Hilbert's distinction between mathematics and metamathematics, the need in general for a distinction between a theory and the corresponding metatheory was recognised. This distinction, which is roughly between the use of a language and speaking about a language, enables us to avoid some of the familiar antinomies. The first contribution to this was made by Lesniewski, but it is here that Tarski made his reputation. Tarski's results in logic and metalogic have been applied by Kuratowski and others in the solution of mathematical problems.

Among the other important contributions of Lukasiewicz is that of many-valued logics. Instead of the classical division of propositions into two, " true " and " false ", in an n-valued logic we divide them into n classes and set up rules to determine the classes of compound propositions.

The second half of the booklet deals with Logical Positivism in Poland ; here the work of Kotarbinski and Ajdukiewicz is outstanding.

The Polish Mathematical and Logical School has lost many distinguished

members in the last six years, but we have been glad to hear that Lukasiewicz and Kuratowski and Tarski (in U.S.A.) are safe. We can therefore hope that their contributions and those of their pupils in the years succeeding the 1939-45 war will be no less important than those made in the years succeeding the last war. O. T.

Reports of a Mathematical Colloquium. 2nd Series. Issues 1–6. $6. (University of Notre Dame Mathematical Publications)

This series of publications, edited by Karl Menger, continues the "Ergebnisse eines Mathematischen Kolloquiums", and is mainly devoted to problems in metric geometry, analysis and topology, to a new development of non-Euclidean geometry, and to the foundation of an algebra of analysis. Two papers on geometry by Artin and Canon Lemaître are included, as well as a contribution to mathematical economics by G. C. Evans. There is also a note on mathematical logic, by the editor.

The volume is attractively printed on good paper. From a perusal of the contents one deduces that mathematics in the U.S.A. was in a flourishing condition until 1942, and has been very little disturbed by three years of war. Furthermore it is clear that the hospitality shown to Karl Menger by the University of Notre Dame has paid handsome cultural dividends to American mathematics. The absorptive capacity of the American continent, the eagerness of its people to welcome distinguished foreigners and to make them feel at home, besides the undoubted talent of American mathematicians, have produced the inevitable result—the standard of American mathematics, at its best, is very high indeed. In time we shall only be able to offer, for the contemplation of those interested in assessing national talents, the spectacle of a few mathematical geniuses, working in isolation, as against an American display of a solid stratum of first-class mathematicians, topped by at least as many geniuses, working in interlocking units.

Something can be done to avoid such a result. I would suggest that, to start with, the older Universities give some mathematical courses in the Long Vacation, with, of course, facilities for residence. This is done, of course, in the States. We, of course, have had "summer schools" for a long time. Why not in mathematics? D. P.

Multi-sensory Aids in the Teaching of Mathematics. Compiled by a Committee of the National Council of Teachers of Mathematics. Eighteenth Year-book. Pp. xv, 455. $2. 1945. (Bureau of Publications, Teachers' College, Columbia University, New York)

To turn over the formidable title-page is to disclose a fascinating collection of devices for gaining attention, interest and understanding in the field of elementary pure mathematics.

About four-fifths of the work is taken up by thirty-six articles by thirty members, whose experience ranges over Junior High Schools, the U.S. Military and Naval Colleges, Teachers' Training Schools, the Massachusetts Institute of Technology, and the Universities. The topics include a general discussion on Visual Aids (by E. R. Breslich), Curve-stitching in Geometry, Mathematical Demonstrations and Exhibits, Linkages in Two and Three Dimensions, Complex Roots, Nomographic Charts, Geometrical Tools, Sources of Information about Early Instruments, Materials, Use of Colour, The Making and Use of Slides, Films and Stereograms, and some excellent Historical material on Models and Mechanical Devices.

The remainder of the book is called an Appendix. The first part contains short illustrated descriptions by their inventors of individual models and devices, suitably classified. The second part is a comprehensive list of

references to periodicals, books, films and filmstrips (English and American) classified under a great many headings. The book is a mine of information, and no teacher could fail to find much useful and stimulating material. The general level of presentation is high, though the title of one article " East High School Vitalizes Mathematics " may cause the lifting of English eyebrows, especially when one reads that the exhibits, financed by the sale of souvenir pencils and magic slates, cost $150, and we may not react favourably to the suggestion that portraits of mathematicians should be framed in " triangles and other geometric figures ".

An introduction by E. H. C. Hildebrandt suggests that most of the credit for the compilation should go to Mary A. Potter and Prof. W. D. Reeve, and in the list of about a hundred and fifty contributors the names of Vera Sanford and H. S. M. Coxeter are familiar to us. Is the use of the feminine Christian name instead of the formal English " Miss " partly responsible for the book's air of sober, youthful, eagerness?

The publishers seem to have been untroubled by " war-economy-standards " : the book is beautifully printed on glossy paper, lavishly illustrated by photograph and diagram, and it stays open anywhere. At two dollars it is a gift, and I hope it will be available over here in sufficient numbers. But open it first and ignore the title : as G. K. Chesterton observed, our American friends are in too great a hurry to think of calling an automobile a car, or an elevator a lift. A. P. R.

Builders' Calculations. By S. H. GLENISTER. Pp. 125. 5s. 1945. (Harrap)

The author, an experienced lecturer, has brought his experience to bear in providing the material for this book.

In a simple way, easy to follow, he has set out by sections a course in Arithmetic dealing with fractions, decimals, square roots, ratio and proportion, squaring and cubing ; a course in Algebra covering the four rules, simple equations, indices and a simplified explanation of the use of logarithms : a course in the geometry and mensuration of the triangle, circle, quadrilateral, the cone, cylinder and the sphere, with a final section showing the use of sine, cosine and tangent.

At the end of each section there is a good selection of practical examples. Whether the reader be one already engaged in the building trade but now seeking knowledge to enable him to undertake estimates of quantities of brick-work, paint-work etc., or an apprentice reading for the First Year National Certificate, or an ex-Service man taking the short-post-war course as a Building Trades trainee, he should benefit very considerably by following through this excellently presented and well-produced book. E. J. A.

An Introduction to the Theory of Numbers. By G. H. HARDY and E. M. WRIGHT. 2nd edition. Pp. xvi, 407. 25s. 1945. (Oxford)

We must be grateful to the Oxford Press for giving us a second edition of this fascinating book (in spite of the difficulties under which publishers still work). It can hardly be necessary to describe it in detail, but it may be said that it gives a full, vivid and exciting account of its subject, as far as this can be done without using too much advanced theory, such as, for instance, the theory of functions of a complex variable. In fact, all the equipment the reader requires is a firm grasp of algebra, of the type that we must now call " old-fashioned ". Granted this, he will learn a great deal about : prime numbers, congruences, irrational numbers, the simpler number fields, Diophantine equations, Diophantine approximation and the " geometry of numbers ". The absence of function-theory does not mean that the book is elementary or devoid of " outlook value ", since it brings us well up to the

present bounds of mathematical knowledge ; Chapter XXIV, " Geometry of Numbers ", for example, is concerned with matters which are very much in activity at the moment, as a reference to recent papers by Davenport, Mordell and others in the publications of the London Mathematical Society will show.

The chief changes from the first edition are those just mentioned in Chapter XXIV, and the new account of the arithmetic of quaternions, where for Dickson's version we now have the slightly more sophisticated but more elegant development given by Hurwitz in a paper in the *Göttinger Nachrichten* and more fully in his *Vorlesungen*. As the changes in Chapter XXIV involved the re-writing of parts of Chapter III, the opportunity might have been used to re-write the initial paragraphs of III, dealing with Farey series ; for, as Professor Neville has remarked to me, the characteristic properties of a Farey series are not at all difficult to establish, being immediate corollaries to a simple and simply-proved result given by him in Note 761 (*Gazette*, XII, p. 343 ; and see Note 1852, p. 24, of this *Gazette*.)

Reading this book reminded me of reading the first volume of Landau's *Primzahlen* : both books compel one to agree that if Mathematics is the Queen of the Sciences, the Theory of Numbers is the Queen of Mathematics.

T. A. A. B.

BOOKS RECEIVED FOR REVIEW.

E. A. Beet. *A textbook of elementary astronomy.* Pp. x, 110. 8s. 6d. 1945. (Cambridge)

R. G. Blaine. *The slide rule.* 6th edition. Pp. xii, 152. 6s. ; postage 4d. 1945. (Spon)

J. L. Coolidge. *A history of the conic sections and quadric surfaces.* Pp. xi, 214. 21s. 1945. (Oxford)

F. Dessauer. *Weltfahrt der Erkenntnis Leben und Werk Isaac Newtons.* Pp. 430. 17·50 fr. (Swiss). 1945. (Rascher Verlag, Zürich)

J. D. H. Donnay. *Spherical trigonometry after the Cesàro method.* Pp. xi, 83. $1.75. 1945. (Interscience Publishers, New York)

G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers.* 2nd edition. Pp. xvi, 407. 25s. 1945. (Oxford)

H. McKay. *The world of numbers.* Pp. vii, 198. 8s. 6d. 1946. (Cambridge)

F. B. Pidduck. *Currents in aerials and high-frequency networks.* Pp. 97. 8s. 6d. 1946. (Oxford)

Bertrand Russell. *Physics and Experience.* Pp. 26. 1s. 6d. 1946. The Henry Sidgwick Lecture, 1945. (Cambridge)

G. Valiron. *Cours d'Analyse mathématique. Théorie des fonctions.* Pp. ii, 522. 300 fr. 1942. *Équations fonctionnelles ; applications.* Pp. ii, 605. 600 fr. 1945. (Masson, Paris)

H. E. Wolfe. *Introduction to non-euclidean geometry.* Pp. xiii, 247. $4.50. 1945. (Dryden Press, New York)

Table of arc sin x. Prepared by the Mathematical Tables Project under the sponsorship of the National Bureau of Standards. Pp. xix, 121. $3.50. 1945. (Columbia University Press, New York)

Tables of associated Legendre functions. Prepared by the Mathematical Tables Project under the sponsorship of the National Bureau of Standards. Pp. xlvi, 303. $5. 1945. (Columbia University Press, New York)

1481. Landsberg is 28 miles south of Königsberg. Bartenstein is 33 miles south-east of Königsberg and 13 miles east of Landsberg.—*The Times*, Feb. 5, 1945. [Per Mr. A. Robson.]

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